On the compatible weakly-nonlocal Poisson brackets of Hydrodynamic Type.

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Abstract

We consider the pairs of general weakly non-local Poisson brackets of Hydrodynamic Type (Ferapontov brackets) and the corresponding integrable hierarchies. We show that under the requirement of non-degeneracy of the corresponding "first" pseudo-Riemannian metric $g_{(0)}^{\nu\mu}$ and also some non-degeneracy requirement for the nonlocal part it is possible to introduce a "canonical" set of "integrable hierarchies" based on the Casimirs, Momentum functional and some "Canonical Hamiltonian functions". We prove also that all the "Higher" "positive" Hamiltonian operators and the "negative" symplectic forms have the weakly non-local form in this case. The same result is also true for "negative" Hamiltonian operators and "positive" Symplectic structures in the case when both pseudo-Riemannian metrics $g_{(0)}^{\nu\mu}$ and $g_{(1)}^{\nu\mu}$ are non-degenerate.

Introduction

We discuss in this paper the Poisson pencils of weakly nonlocal Poisson brackets of Hydrodynamic Type (Ferapontov brackets). This means that we consider a pair of Hamiltonian operators:

$$\hat{J}_{(0)}^{\nu\mu} = g_{(0)}^{\nu\mu}(U)\frac{d}{dX} + b_{(0)\eta}^{\nu\mu}(U)U_X^\eta + \sum_{k=1}^{g_0} e_{(0)k} w_{(0)k\eta}^\nu(U)U_X^\eta D^{-1} w_{(0)k\zeta}^\mu(U)U_X^\zeta$$

$$\hat{J}_{(1)}^{\nu\mu} = g_{(1)}^{\nu\mu}(U)\frac{d}{dX} + b_{(1)\eta}^{\nu\mu}(U)U_X^\eta + \sum_{k=1}^{g_1} e_{(1)k} w_{(1)k\eta}^\nu(U)U_X^\eta D^{-1} w_{(1)k\zeta}^\mu(U)U_X^\zeta$$

where $e_{(0)k}, e_{(1)k} = \pm 1$ and $D^{-1} = (d/dX)^{-1}$ defined in a "skew-symmetric" way:

$$D^{-1} = \frac{1}{2}\left[\int_{-\infty}^X dX - \int_X^{+\infty} dX\right]$$

and require that the expression

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$$\hat{J}_\lambda^{\nu\mu} = \hat{J}_{(0)}^{\nu\mu} + \lambda \hat{J}_{(1)}^{\nu\mu}$$

defines the Poisson bracket satisfying Jacobi identity for any λ.

Let us mention here that the brackets of this kind are the generalization of Dubrovin - Novikov local homogeneous brackets of Hydrodynamic Type ([1]-[3]):

$$\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(U)\delta'(X-Y) + b_\eta^{\nu\mu}(U)U_X^\eta\delta(X-Y) \qquad (0.1)$$

with the Hamiltonian operator:

$$\hat{J}_{DN}^{\nu\mu} = g^{\nu\mu}(U)\frac{d}{dX} + b_\eta^{\nu\mu}(U)U_X^\eta$$

Theorem. (B.A.Dubrovin, S.P.Novikov)

Consider the bracket (0.1) with non-degenerate tensor $g^{\nu\mu}(U)$. From the Leibnitz identity it follows that $g^{\nu\mu}(U)$ and $\Gamma_{\nu\eta}^\mu(U) = -g_{\nu\xi}(U)b_\eta^{\xi\mu}(U)$ $(g_{\nu\xi}g^{\xi\mu} = \delta_\nu^\mu)$ transform as a metric with upper indices and the Christoffel symbols under the pointwise coordinate transformations $\tilde{U}^\nu = \tilde{U}^\nu(U)$.

The bracket (0.1) is skew-symmetric if and only if $g^{\nu\mu}$ is symmetric and the connection $\Gamma_{\nu\eta}^\mu$ is compatible with the metric: $\nabla_\eta g^{\nu\mu} \equiv 0$.

The bracket (0.1) satisfies the Jacobi identity if and only if the connection $\Gamma_{\nu\eta}^\mu$ is symmetric and has zero curvature: $R_{\eta\xi}^{\nu\mu} \equiv 0$.

It follows from Dubrovin - Novikov theorem that any bracket (0.1) with non-degenerate $g^{\nu\mu}$ can be written locally in the "constant form":

$$\{n^\nu(X), n^\mu(Y)\} = \epsilon^\nu \delta^{\nu\mu} \delta'(X-Y) , \quad \epsilon^\nu = \pm 1 \qquad (0.2)$$

in the flat coordinates $n^\nu = n^\nu(U)$.

The functionals

$$N^\nu = \int_{-\infty}^{+\infty} n^\nu(X)dX$$

are Casimirs of the bracket (0.1) and the functional

$$P = \int_{-\infty}^{+\infty} \frac{1}{2} \sum_{\nu=1}^{N} \epsilon^\nu n^\nu(X)n^\nu(X)dX$$

is a momentum operator generating the flow $U_T^\nu = U_X^\nu$. The form (0.2) can be considered as the Canonical form for the DN-bracket (0.1) with the non-degenerate tensor $g^{\nu\mu}$.

It can be seen also that any functional of "Hydrodynamic Type"

$$H = \int_{-\infty}^{+\infty} h(U)dX$$

generates a "Hydrodynamic Type System"

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$$U_T^\nu = V_\mu^\nu(U)U_X^\mu \tag{0.3}$$

according to bracket (0.1).

Let us mention also that the brackets (0.1) with degenerate tensor $g^{\nu\mu}(U)$ of constant rank has more complicated but also nice differential geometric structure (see [4]).

The first generalization of DN-bracket to the weakly nonlocal case was the Mokhov-Ferapontov bracket [5]:

$$\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(U)\delta'(X-Y) + b_\eta^{\nu\mu}(U)U_X^\eta\delta(X-Y) + cU_X^\nu\nu(X-Y)U_Y^\mu \tag{0.4}$$

where $\nu(X-Y) = 1/2\,sgn(X-Y)$, corresponding to the Hamiltonian operator:

$$\hat{J}_{DN}^{\nu\mu} = g^{\nu\mu}(U)\frac{d}{dX} + b_\eta^{\nu\mu}(U)U_X^\eta + cU_X^\nu D^{-1}U_X^\mu$$

Theorem. (O.I.Mokhov, E.V.Ferapontov).

Consider the bracket (0.4) with non-degenerate tensor $g^{\nu\mu}(U)$. Then:

1) The bracket (0.4) is skew-symmetric and satisfies Leibnitz identity if and only if tensor $g^{\nu\mu}(U)$ is a metric with upper indices and $\Gamma_{\nu\eta}^\mu = -g_{\nu\xi}b_\eta^{\xi\mu}$ are the connection coefficients compatible with $g^{\nu\mu}(U)$.

2) The bracket (0.4) satisfies the Jacobi identity if and only if the connection $\Gamma_{\nu\eta}^\mu$ is symmetric and has the constant curvature equal to c, i.e.

$$R_{\mu\eta}^{\nu\tau} = c\left(\delta_\mu^\nu\delta_\eta^\tau - \delta_\mu^\tau\delta_\eta^\nu\right)$$

The bracket (0.4) has a weakly nonlocal form. However, any local translationally invariant functional

$$H = \int h(U)dX$$

generates a local system of Hydrodynamic Type with respect to (0.4). Indeed, we have

$$U_X^\mu\frac{\partial h}{\partial U^\mu} \equiv \partial_X h$$

if h does not depend on X explicitly, so the application of D^{-1} gives the local expression for the corresponding flow.

The Canonical form of the bracket (0.4) was presented first by M.V.Pavlov in [6] and can be written as:

$$\{n^\nu(X), n^\mu(Y)\} =$$

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$$= \left(\epsilon^{\nu}\delta^{\nu\mu} - cn^{\nu}n^{\mu}\right)\delta'(X-Y) - cn_X^{\nu}n^{\mu}\delta(X-Y) + cn_X^{\nu}\nu(X-Y)n_Y^{\mu} \tag{0.5}$$

where $n^{\nu} = n^{\nu}(U)$ are the annihilators for the bracket (0.4) (on the space of rapidly decreasing functions $n^{\nu}(X)$ at $X \to \pm\infty$). Also the implicit expression for the density of P was represented in [6].

We will see however that the Casimirs and the Momentum operator for the bracket (0.4) depend actually on the boundary conditions imposed on the functions $U^{\nu}(X)$ for $X \to \pm\infty$ ([19]) (the condition $U^{\nu} \to 0$, $X \to \pm\infty$ in general is not invariant under the pointwise transformations $\tilde{U}^{\nu} = \tilde{U}^{\nu}(U)$). As was pointed out in [19] we can not speak about the Casimirs and Momentum functional until we fix the boundary conditions at infinity and in the general case it is better to speak about the invariant set of $N+1$ (for MF-bracket) functionals playing the role of either Casimirs of Momentum operator according to the boundary conditions. Let us consider this later for the case of more general Ferapontov brackets.

The general Ferapontov bracket ([7]-[10]) has the form:

$$\{U^{\nu}(X), U^{\mu}(Y)\} = g^{\nu\mu}(U)\delta'(X-Y) + b_{\eta}^{\nu\mu}(U)U_X^{\eta}\delta(X-Y)+$$

$$+ \sum_{k=1}^{g} e_k w_{k\eta}^{\nu}(U)U_X^{\eta}\nu(X-Y)w_{k\zeta}^{\mu}(U)U_Y^{\zeta} \tag{0.6}$$

$e_k = \pm 1$, which corresponds to the weakly nonlocal Hamiltonian operator:

$$\hat{J}_F^{\nu\mu} = g^{\nu\mu}(U)\frac{d}{dX} + b_{\eta}^{\nu\mu}(U)U_X^{\eta} + \sum_{k=1}^{g} e_k w_{k\eta}^{\nu}(U)U_X^{\eta}D^{-1}w_{k\zeta}^{\mu}(U)U_X^{\zeta} \tag{0.7}$$

Theorem. (E.V.Ferapontov).
Consider the bracket (0.6) with non-degenerate tensor $g^{\nu\mu}(U)$. Then:

1) The bracket (0.6) is skew-symmetric and satisfies Leibnitz identity if and only if tensor $g^{\nu\mu}(U)$ is a metric with upper indices and $\Gamma_{\nu\eta}^{\mu} = -g_{\nu\xi}b_{\eta}^{\xi\mu}$ are the connection coefficients compatible with $g^{\nu\mu}(U)$.

2) The bracket (0.4) satisfies the Jacobi identity if and only if the connection $\Gamma_{\nu\eta}^{\mu}$ is symmetric and the metric $g^{\nu\mu}$ and tensors $w_{k\eta}^{\nu}(U)$ satisfy the equations:

$$g^{\nu\tau}w_{k\tau}^{\mu} = g^{\mu\tau}w_{k\tau}^{\nu} , \quad \nabla_{\nu}w_{k\eta}^{\mu} = \nabla_{\eta}w_{k\nu}^{\mu}$$

$$R_{\mu\eta}^{\nu\tau} = \sum_{k=1}^{g} e_k \left(w_{k\mu}^{\nu}w_{k\eta}^{\tau} - w_{k\mu}^{\tau}w_{k\eta}^{\nu}\right)$$

Moreover, this set is commutative $[w_k, w_{k'}] = 0$.

It was pointed out by E.V.Ferapontov that the equations written above are Gauss and Petersson-Codazzi equations for the submanifold $\mathcal{M}^N$ with flat normal connection in the pseudo-Euclidean space E^{N+g}. In this consideration tensor $g^{\nu\mu}$ is the first quadratic form

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of $\mathcal{M}^N$ and $w_{k\eta}^\nu$ are the Weingarten operators corresponding to g parallel vector fields in the normal bundle $\mathbf{N}_k$, such that $\langle \mathbf{N}_k, \mathbf{N}_m \rangle = e_k \delta_{km}$. It was also proved by E.V.Ferapontov that these brackets can be constructed as a Dirac restriction of the local DN-bracket

$$\{Z^I(X), Z^J(Y)\} = \epsilon^I \delta^{IJ} \delta'(X - Y) \ , \ \ I, J = 1, \dots, N + g, \ \ \epsilon^I = \pm 1$$

in E^{N+g} to the submanifold $\mathcal{M}^N$ ([8],[10]).

As far as we know the cases of brackets (0.4), (0.6) with the degenerate tensors $g^{\nu\mu}(U)$ were not studied in the literature.

All the brackets (0.1), (0.4) and (0.6) are closely connected with the diagonalizable integrable systems (0.3).

The general procedure of integration of the so-called "semi-Hamiltonian" diagonal systems of Hydrodynamic Type was constructed by S.P.Tsarev ([11], [12]). It can be shown that any diagonal system (0.3) which is Hamiltonian with respect to bracket (0.1), (0.4) or (0.6) (with diagonal $g^{\nu\mu}(U)$) satisfies Tsarev "semi-Hamiltonian" property and so can be integrated by Tsarev's method. Probably, all semi-Hamiltonian systems are in fact Hamiltonian corresponding to some weakly nonlocal H.T.P.B. with (maybe) an infinite number of terms in the nonlocal tail. Some investigation of this problem can be found in [10], [13] but in general this problem is still open. Let us mention also that the examples of non-diagonalizable Hamiltonian integrable (by inverse scattering methods) systems (0.3) were also investigated in [14]-[15].

As was pointed out in [9]-[10], if the manifold $\mathcal{M}^N$ has a holonomic net of lines of curvature the metric $g^{\nu\mu}(U)$ and all the operators $w_{k\eta}^\nu$ can be written in the diagonal form in the corresponding coordinates $r^\nu = r^\nu(U)$. Here we don't impose this requirement and consider any brackets of Ferapontov type.

We will assume that the flows $w_{k\eta}^\nu(U)U_X^\eta$ in the nonlocal part of (0.6) are linearly independent (with constant coefficients). (The nonlocal part in (0.6) represents actually the non-degenerate quadratic form on the linear space generated by $w_{k\eta}^\nu(U)U_X^\eta$, $k = 1, \dots, g$ written in the canonical form with $e_k = \pm 1$.) As was pointed out by E.V.Ferapontov the local functional

$$H = \int h(U) dX$$

generates in this case the local flow with respect to the bracket (0.6) if and only if the functional H is a conservation law for any of the flows $w_{k\eta}^\nu(U)U_X^\eta$ such that the expressions

$$w_{k\eta}^\nu(U)U_X^\eta \frac{\partial h}{\partial U^\nu}$$

represent the total derivatives with respect to X of some functions $Q_k(U)$ for any k.

This fact is also true for more general weakly nonlocal Poisson brackets having the form:

$$\{\varphi^i(x), \varphi_j(y)\} = \sum_{k=1}^{G} B_k^{ij}(\varphi, \varphi_x, \dots)\delta^{(k)}(x - y) +$$

$$+ \sum_{k=1}^{g} e_k S_k^i(\varphi, \varphi_x, \dots)\nu(x - y)S_k^j(\varphi, \varphi_y, \dots) \tag{0.8}$$

where $\delta^{(k)}(x-y) = (d/dx)^k \delta(x-y)$, $e_k = \pm 1$, and the set $\{S_k^i(\varphi, \varphi_x, \ldots)\}$ is linearly independent.

As far as we know the first example written precisely in this form was the Sokolov bracket ([16])

$$\{\varphi(x), \varphi(y)\} = -\varphi_x \nu(x-y) \varphi_y$$

for the Krichever-Novikov equation

$$\varphi_t = \varphi_{xxx} - \frac{3}{2} \frac{\varphi_{xx}^2}{\varphi_x} + \frac{h(\varphi)}{\varphi_x}$$

where $h(\varphi) = c_3 \varphi^3 + c_2 \varphi^2 + c_1 \varphi + c_0$, with the Hamiltonian function

$$H = \int \left(\frac{1}{2} \frac{\varphi_{xx}^2}{\varphi_x^2} + \frac{1}{3} \frac{h(\varphi)}{\varphi_x^2} \right) dx$$

As was established in [27]-[28] the flows $S_k^i(\varphi, \varphi_x, \ldots)$ commute with each other for any general bracket (0.8) and conserve the corresponding Hamiltonian structure (0.8) on the phase space $\{\varphi^i(x)\}$ (this fact was important for the averaging procedure for such brackets considered there). However, for general brackets (0.8) they are not necessarily generated by the local Hamiltonian functions having the form

$$H = \int h(\varphi, \varphi_x, \ldots) dx$$

Actually the brackets (0.8) are very common for so-called "integrable systems" (like KdV or NLS) possessing the multi-Hamiltonian structures connected by the recursion operator according to the Lenard-Magri scheme ([17]). Such, it was proved in [18] that all the higher PB brackets for KdV given by the recursion scheme starting from Gardner-Zakharov-Faddev bracket

$$\{\varphi(x), \varphi(y)\} = \delta'(x-y)$$

and the Magri bracket

$$\{\varphi(x), \varphi(y)\} = -\delta'''(x-y) + 4\varphi(x)\delta'(x-y) + 2\varphi_x \delta(x-y)$$

have exactly the form (0.8). In [19] the same fact was proved for the case of NLS hierarchy and also the weakly nonlocal form of the "negative" symplectic forms for KdV and NLS was established.

The brackets (0.1), (0.4) and (0.6) appear in these systems as the "dispersionless" limit of the corresponding bracket (0.8) or, in more general case, as a result of the averaging of (0.8) on the families of quasiperiodic solutions of corresponding evolution system connected with the Whitham method for slow modulations of parameters ([1]-[3], [20]-[28]).

We consider here the compatible brackets of Ferapontov type and prove the similar facts for the case of the non-degenerate pencils (i.e. $det\, g_{(0)}^{\nu\mu} \neq 0$) with also some non-degeneracy conditions for non-local part of $\hat{J}_{(0)} + \lambda \hat{J}_{(1)}$.

Let us mention also that the wide classes of local pencils of Hydrodynamic Type (DN-brackets) were investigated in many details in [29] (see also [30], [31] and references therein) where they play the important role in the structure of Dubrovin-Frobenius manifolds connected with solutions of WDVV equation for Topological Field Theories. In the works [10], [32] some important questions of weakly non-local pencils of Hydrodynamic Type (H.T.) were also considered. In the papers [33], [34] also the generic diagonal compatible flat pencils in terms of inverse scattering method (see [35], [36]) were discussed.

1 On the canonical form and symplectic operator for the general F-bracket.

Let us formulate now the properties of the brackets (0.6) established in [19] which we will need in further consideration.

Let us consider the bracket (0.6) with non-degenerate tensor $g^{\nu\mu}(U)$. According to Ferapontov results we can represent it as a Dirac restriction of DN-bracket in R^{N+g} to the submanifold $\mathcal{M}^N \subset R^{N+g}$ with flat normal connection. Let us fix the point $z \in \mathcal{M}^N$ and introduce the corresponding loop space in the vicinity of z:

$$L(\mathcal{M}^N, z) = \{\gamma : R^1 \to \mathcal{M}^N : \gamma(-\infty) = \gamma(+\infty) = z \in \mathcal{M}^N\}$$

So we fix the boundary conditions for the functions $U^\nu(X)$ and require that these functions rapidly approach their boundary values at $X \to \pm\infty$.

Theorem. (A.Ya.Maltsev, S.P.Novikov)
Let us consider the bracket (0.6) with non-degenerate $g^{\nu\mu}(U)$ defined on the loop space $L(\mathcal{M}^N, z)$. Consider the corresponding embedding $\mathcal{M}^N \subset R^{N+g}$ with flat normal connection. Consider the flat orthogonal coordinates Z^I in R^{N+g} such that:
1) $Z^I(z) = 0$, $I = 1, \ldots, N + g$ and the corresponding DN-bracket in R^{N+g} has the form

$$\{Z^I(X), Z^J(Y)\} = E^I \delta^{IJ} \delta'(X - Y) , \quad I, J = 1, \ldots, N + g , \quad E^I = \pm 1$$

2) The first N coordinates $Z^1, \ldots, Z^N$ are tangential to $\mathcal{M}^N$ at the point z.
3) The last g coordinates $Z^{N+1}, \ldots, Z^{N+g}$ are orthogonal to $\mathcal{M}^N$ at the point z.
Then:
1) The bracket (0.6) has exactly N local Casimirs given by the functionals

$$N^\nu = \int_{-\infty}^{+\infty} n^\nu(X) dX$$

where $n^\nu(U)$ are the restrictions of the first N (tangential to $\mathcal{M}^N$ at z) coordinates $Z^1, \ldots, Z^N$ on $\mathcal{M}^N$.
2) All the flows

$$U_{t_k}^\nu = w_{k\eta}^\nu(U) U_X^\eta$$

are Hamiltonian with respect to (0.6) with local Hamiltonian functionals

$$H_k = \int_{-\infty}^{+\infty} h_k(X)dX \ , \ \ k = 1, \ldots, g$$

where $h_k(U)$ are the restriction of the last g coordinates Z^{N+k}, $k = 1, \ldots, g$ on $\mathcal{M}^N$ and $w_{k\eta}^\nu(U)$ are the Weingarten operators corresponding to parallel vector fields $\mathbf{N}_k(U)$ in the normal bundle with the normalization:

$$\mathbf{N}_k(z) = (0, \ldots, 0, -E^{N+k}, 0, \ldots, 0)^T$$

(where $-E^{N+k}$ stays at the position $N+k$) in the coordinates $Z^1, \ldots, Z^{N+g}$ (so $\langle \mathbf{N}_k(U), \mathbf{N}_l(U) \rangle = E^{N+k}\delta_{kl}$).[1]

3) The bracket (0.6) has in coordinates $n^\nu(U)$ the "Canonical form" corresponding to the space $L(\mathcal{M}^N, z)$, i.e.

$$\{n^\nu(X), n^\mu(Y)\} = \left(\epsilon^\nu \delta^{\nu\mu} - \sum_{k=1}^{g} e_k f_k^\nu(n) f_k^\mu(n) \right) \delta'(X - Y) -$$

$$- \sum_{k=1}^{g} e_k \left(f_k^\nu(n) \right)_X f_k^\mu(n)\delta(X - Y) + \sum_{k=1}^{g} e_k \left(f_k^\nu(n) \right)_X \nu(X - Y) \left(f_k^\mu(n) \right)_Y$$

where $\epsilon^\nu = E^\nu$, $\nu = 1, \ldots, N$, $e_k = E^{N+k}, k = 1, \ldots, g$, $n^\nu(z) = 0$, $f_k^\nu(z) = 0$ (i.e. $f_k^\nu(0) = 0$). (Actually we have the equality $f_k^\nu(U) \equiv N_k^\nu(U)$, where $N_k^\nu(U)$ are the first N components of the vectors $\mathbf{N}_k(U)$ in the coordinates $Z^1, \ldots, Z^{N+g}$).

Let us note here that the Casimirs and "Canonical functionals", as well as the Canonical forms depend on the phase space $L(\mathcal{M}^N, z)$. So, if we don't fix the loop space it is better to speak just about the $N + g$ canonical functions (the restrictions of the flat coordinates from R^{N+g}) playing the role of Casimirs or Canonical functionals depending on the boundary conditions. (Also we will have many "Canonical forms" of the bracket (0.6) with different $f_k^\nu(U)$ in this case).

For the case of Mokhov-Ferapontov bracket the Canonical form will be the (0.5) for any fixed space $L(\mathcal{M}^N, z)$ (although the coordinates $n^\nu(U)$ will be different for different loop spaces). The explicit form of canonical functional, which is the momentum operator in this case, can be written then as

$$P = \frac{1}{c} \int_{-\infty}^{+\infty} \left(1 - \sqrt{1 - c \sum_{\nu=1}^{N} \epsilon^\nu n^\nu n^\nu} \right) dX$$

([19]).

Using the restriction of the symplectic form of DN-bracket on $\mathcal{M}^N$ it is also possible to get the symplectic form $\Omega_{\nu\mu}(X, Y)$ for the F-bracket (0.6) with non-degenerate $g^{\nu\mu}(U)$ ([19]). The symplectic form appears to be also weakly nonlocal and can be written as

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[1]There is an arithmetic mistake in the journal variant of [19] where the generated flows are written as $E^{N+k}w_{k\eta}^\nu(U)U_X^\eta$.

$$\Omega_{\nu\mu}(X,Y) = \sum_{I=1}^{N+g} E^I \frac{\partial Z^I(U)}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial Z^I(U)}{\partial U^\mu}(Y) =$$

$$= \sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial n^\tau}{\partial U^\mu}(Y) + \sum_{k=1}^{g} e_k \frac{\partial h_k}{\partial U^\nu}(X)\nu(X-Y)\frac{\partial h_k}{\partial U^\mu}(Y)$$

on $\mathcal{M}^N$.

We can write also the corresponding symplectic operator $\hat{\Omega}_{\nu\mu}$ on $L(\mathcal{M}^N, z)$ as

$$\hat{\Omega}_{\nu\mu} = \sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu} D^{-1} \frac{\partial n^\tau}{\partial U^\mu} + \sum_{k=1}^{g} e_k \frac{\partial h_k}{\partial U^\nu} D^{-1} \frac{\partial h_k}{\partial U^\mu} \tag{1.1}$$

with D^{-1} defined in the skew-symmetric way.

Let us introduce the functional space $\mathcal{V}_0(z)$ of the vector-fields $\xi^\nu(X)$ on $L(\mathcal{M}^N, z)$ rapidly decreasing $\xi^\nu(X) \to 0$ at $X \to \pm\infty$. It is easy to see that all the Hydrodynamic Type systems (0.3) satisfy this requirement as the vector fields on $L(\mathcal{M}^N, z)$ and so belong to $\mathcal{V}_0(z)$. We can then naturally define the action of symplectic operator $\hat{\Omega}_{\nu\mu}$ on the space $\mathcal{V}_0(z)$.

We will need also the functional space $\mathcal{F}_0(z)$ of 1-forms $\omega_\nu(X)$ on $L(\mathcal{M}^N, z)$ such that $\omega_\nu(X) \to 0$ (rapidly decreasing) at $X \to \pm\infty$. Let us note here that $\hat{\Omega}_{\nu\mu}\xi^\mu(X) \notin \mathcal{F}_0(z)$ in the general case.

We will prove now by the direct calculation that the symplectic form $\hat{\Omega}_{\nu\mu}$ is the inverse of the Hamiltonian operator $\hat{J}_F^{\nu\mu}$ on the appropriate functional spaces.

Theorem 1.
(I) We have on the functional space $\mathcal{V}_0(z)$ the relation

$$\hat{J}_F^{\nu\xi}\hat{\Omega}_{\xi\mu} = \hat{I}$$

where $\hat{I}$ is the identity operator on the space $\mathcal{V}_0(z)$.

(II) We have on the functional space $\mathcal{F}_0(z)$:

$$\hat{\Omega}_{\nu\xi}\hat{J}_F^{\xi\mu} = \hat{I}$$

where $\hat{I}$ is the identity operator on $\mathcal{F}_0(z)$.

Proof.
(I) We have

$$\hat{J}_F^{\nu\xi}\hat{\Omega}_{\xi\mu} = \left[g^{\nu\xi}(U)\frac{d}{dX} + b_\eta^{\nu\xi}(U)U_X^\eta + \sum_{k=1}^{g} e_k w_{k\eta}^\nu(U)U_X^\eta D^{-1} w_{k\zeta}^\xi(U)U_X^\zeta \right] \times$$

$$\times \left[\sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\xi} D^{-1} \frac{\partial n^\tau}{\partial U^\mu} + \sum_{l=1}^{g} e_l \frac{\partial h^l}{\partial U^\xi} D^{-1} \frac{\partial h^l}{\partial U^\mu} \right]$$

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where the operators $w_{k\eta}^\nu(U)$ correspond to the vector fields $\mathbf{N}_{(k)}(U)$ such that

$$\mathbf{N}_{(k)}(z) = (0, \ldots, 0, -e_k, 0, \ldots, 0)^T \tag{1.2}$$

($-e_k$ stays at the position $N+k$) and we have identically $\langle \mathbf{N}_{(k)}(U), \mathbf{N}_{(l)}(U) \rangle = e_k \delta_{kl}$.

Since the integrals of $n^\tau(U)$ and $h^l(U)$ generate the local flows according to $\hat{J}_F^{\nu\mu}$ we have:

$$w_{k\zeta}^\xi(U) U_X^\zeta \frac{\partial n^\tau}{\partial U^\xi} \equiv (q_k^\tau(U))_X \ , \quad w_{k\zeta}^\xi(U) U_X^\zeta \frac{\partial h^l}{\partial U^\xi} \equiv \left(p_k^l(U)\right)_X \tag{1.3}$$

for some functions $q_k^\tau(U)$ and $p_k^l(U)$. Moreover, let us consider N tangential vectors to $\mathcal{M}_N$ in R^{N+g} corresponding to the coordinates U^ν:

$$\mathbf{e}_{(\xi)}(U) = \left(\frac{\partial n^1}{\partial U^\xi}, \ldots, \frac{\partial n^N}{\partial U^\xi}, \frac{\partial h^1}{\partial U^\xi}, \ldots, \frac{\partial h^g}{\partial U^\xi} \right)^T \tag{1.4}$$

and our parallel orthogonal vector fields $\mathbf{N}_{(k)}(U)$ defined by the condition (1.2) in the coordinates $(Z^1, \ldots, Z^{N+g})$;

Since by the definition:

$$\frac{d}{dX} \mathbf{N}_{(k)}(U) = w_{k\zeta}^\xi U_X^\zeta \mathbf{e}_{(\xi)}(U)$$

we have from (1.3)

$$(q_k^\tau)_X = \left(N_{(k)}^\tau \right)_X \ , \quad \left(p_k^l \right)_X = \left(N_{(k)}^{N+l} \right)_X \ , \quad \tau = 1, \ldots, N \ , \quad l = 1, \ldots g$$

for the components of $\mathbf{N}_{(k)}(U)$.

Let us normalize the functions $q_k^\tau(U)$ and $p_k^l(U)$ such that

$$q_k^\tau(z) = 0 \ , \quad p_k^l(z) = 0 \tag{1.5}$$

We have then

$$q_k^\tau(U) = N_{(k)}^\tau(U) \ , \quad p_k^l(U) = N_{(k)}^{N+l}(U) + e_k \delta_k^l \tag{1.6}$$

Now using the equalities

$$(q_k^\tau)_X = \frac{d}{dX} q_k^\tau - q_k^\tau \frac{d}{dX} \ , \quad \left(p_k^l \right)_X = \frac{d}{dX} p_k^l - p_k^l \frac{d}{dX}$$

on $L(\mathcal{M}^N, z)$ we can write

$$\hat{J}_F^{\nu\xi} \hat{\Omega}_{\xi\mu}|_{\mathcal{V}_0(z)} =$$

$$= \sum_{\tau=1}^N \epsilon^\tau \left[g^{\nu\xi} \left(\frac{\partial n^\tau}{\partial U^\xi} \right)_X + b_\eta^{\nu\xi} U_X^\eta \frac{\partial n^\tau}{\partial U^\xi} + \sum_{k=1}^g e_k w_{k\eta}^\nu U_X^\eta D^{-1} \frac{d}{dX} q_k^\tau \right] D^{-1} \frac{\partial n^\tau}{\partial U^\mu} +$$

$$+\sum_{l=1}^{g} e_l \left[g^{\nu\xi} \left(\frac{\partial h^l}{\partial U^\xi} \right)_X + b_\eta^{\nu\xi} U_X^\eta \frac{\partial h^l}{\partial U^\xi} + \sum_{k=1}^{g} e_k w_{k\eta}^\nu U_X^\eta D^{-1} \frac{d}{dX} p_k^l \right] D^{-1} \frac{\partial h^l}{\partial U^\mu} +$$

$$+ g^{\nu\xi} \left[\sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\xi} \frac{d}{dX} D^{-1} \frac{\partial n^\tau}{\partial U^\mu} + \sum_{l=1}^{g} e_l \frac{\partial h^l}{\partial U^\xi} \frac{d}{dX} D^{-1} \frac{\partial h^l}{\partial U^\mu} \right] -$$

$$- \sum_{k=1}^{g} \sum_{\tau=1}^{N} e_k \epsilon^\tau w_{k\eta}^\nu U_X^\eta D^{-1} q_k^\tau \frac{d}{dX} D^{-1} \frac{\partial n^\tau}{\partial U^\mu} - \sum_{k=1}^{g} \sum_{l=1}^{g} e_k e_l w_{k\eta}^\nu U_X^\eta D^{-1} p_k^l \frac{d}{dX} D^{-1} \frac{\partial h^l}{\partial U^\mu}$$

We can replace here $(d/dX)D^{-1}$ by identity and we have

$$\left(D^{-1} f_X \right) = f(X) - \frac{1}{2} \left[f(-\infty) + f(+\infty) \right]$$

for any function $f(U)$ on $L(\mathcal{M}^N, z)$ according to the definition of D^{-1}. So, according to normalization (1.5) we can replace the operators $D^{-1}(d/dX)q_k^\tau$ and $D^{-1}(d/dX)p_k^l$ just by q_k^τ and p_k^l on $L(\mathcal{M}^N, z)$.

According to the same normalization the expressions within the brackets in the first two terms are equal to

$$\left[\hat{J}_F^{\nu\xi} \frac{\partial n^\tau}{\partial U^\xi} \right]_{L(\mathcal{M}^N, z)} = 0 \quad \text{and} \quad \left[\hat{J}_F^{\nu\xi} \frac{\partial h^l}{\partial U^\xi} \right]_{L(\mathcal{M}^N, z)} = w_{l\eta}^\nu U_X^\eta \tag{1.7}$$

So we have

$$\hat{J}_F^{\nu\xi} \hat{\Omega}_{\xi\mu} |_{v_0(z)} = \sum_{l=1}^{g} e_l w_{l\eta}^\nu U_X^\eta D^{-1} \frac{\partial h^l}{\partial U^\mu} + g^{\nu\xi} \left[\sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\xi} \frac{\partial n^\tau}{\partial U^\mu} + \sum_{l=1}^{g} e_l \frac{\partial h^l}{\partial U^\xi} \frac{\partial h^l}{\partial U^\mu} \right] -$$

$$- \sum_{k=1}^{g} e_k w_{k\eta}^\nu U_X^\eta D^{-1} \left[\sum_{\tau=1}^{N} \epsilon^\tau q_k^\tau \frac{\partial n^\tau}{\partial U^\mu} + \sum_{l=1}^{g} e_l p_k^l \frac{\partial h^l}{\partial U^\mu} \right]$$

Using (1.6) and (1.4) we get now

$$\sum_{\tau=1}^{N} \epsilon^\tau q_k^\tau \frac{\partial n^\tau}{\partial U^\mu} + \sum_{l=1}^{g} e_l p_k^l \frac{\partial h^l}{\partial U^\mu} = \langle \mathbf{N}_{(k)}, \mathbf{e}_{(\mu)} \rangle + \frac{\partial h^k}{\partial U^\mu} = \frac{\partial h^k}{\partial U^\mu} \tag{1.8}$$

Using also the evident relation

$$\sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\xi} \frac{\partial n^\tau}{\partial U^\mu} + \sum_{l=1}^{g} e_l \frac{\partial h^l}{\partial U^\xi} \frac{\partial h^l}{\partial U^\mu} \equiv g_{\xi\mu}(U)$$

we get the statement (I) of the theorem.

(II) We have

$$\hat{\Omega}_{\nu\xi} \hat{J}_F^{\xi\mu} = \left(\sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu} D^{-1} \frac{\partial n^\tau}{\partial U^\xi} + \sum_{l=1}^{g} e_l \frac{\partial h^l}{\partial U^\nu} D^{-1} \frac{\partial h^l}{\partial U^\xi} \right) \times$$

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$$\times \left(g^{\xi\mu}\frac{d}{dX} + b_\eta^{\xi\mu}U_X^\eta + \sum_{k=1}^{g} e_k w_{k\eta}^\xi U_X^\eta D^{-1}w_{k\zeta}^\mu U_X^\zeta \right) =$$

$$= \sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu} D^{-1}\frac{d}{dX}\frac{\partial n^\tau}{\partial U^\xi}g^{\xi\mu} + \sum_{l=1}^{g} \frac{\partial h^l}{\partial U^\nu} D^{-1}\frac{d}{dX}\frac{\partial h^l}{\partial U^\xi}g^{\xi\mu} -$$

$$- \sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu} D^{-1}\left(\frac{\partial n^\tau}{\partial U^\xi}g^{\xi\mu}\right)_X - \sum_{l=1}^{g} \frac{\partial h^l}{\partial U^\nu} D^{-1}\left(\frac{\partial h^l}{\partial U^\xi}g^{\xi\mu}\right)_X +$$

$$+ \sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu} D^{-1}\frac{\partial n^\tau}{\partial U^\xi}b_\eta^{\xi\mu}U_X^\eta + \sum_{l=1}^{g} e_l \frac{\partial h^l}{\partial U^\nu} D^{-1}\frac{\partial h^l}{\partial U^\xi}b_\eta^{\xi\mu}U_X^\eta +$$

$$+ \sum_{\tau=1}^{N}\sum_{k=1}^{g} \epsilon^\tau e_k \frac{\partial n^\tau}{\partial U^\nu} D^{-1}\left(\frac{d}{dX}q_k^\tau - q_k^\tau\frac{d}{dX}\right)D^{-1}w_{k\zeta}^\mu U_X^\zeta +$$

$$+ \sum_{l=1}^{g}\sum_{k=1}^{g} e_l e_k \frac{\partial h^l}{\partial U^\nu} D^{-1}\left(\frac{d}{dX}p_k^l - p_k^l\frac{d}{dX}\right)D^{-1}w_{k\zeta}^\mu U_X^\zeta$$

We can replace again the operators $(d/dX)D^{-1}$ by identity and $D^{-1}(d/dX)q_k^\tau$ and $D^{-1}(d/dX)p_k^l$ by q_k^τ and p_k^l. Then according to the definition of coordinates h^l we have $(\partial h^l/\partial U^\xi)(z) = 0$, so we put also $D^{-1}(d/dX)(\partial h^l/\partial U^\xi) = (\partial h^l/\partial U^\xi)$. We get now using the same arguments

$$\hat{\Omega}_{\nu\xi}\hat{J}_F^{\xi\mu} = \hat{I} + \sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu}\left[D^{-1}\frac{d}{dX} - \hat{I}\right]\frac{\partial n^\tau}{\partial U^\xi}g^{\xi\mu} -$$

$$- \sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu} D^{-1}\left[\hat{J}_F^{\mu\xi}\frac{\partial n^\tau}{\partial U^\xi}\right]_{L(\mathcal{M}^N,z)} - \sum_{l=1}^{g} e_l \frac{\partial h^l}{\partial U^\nu} D^{-1}\left[\hat{J}_F^{\mu\xi}\frac{\partial h^l}{\partial U^\xi}\right]_{L(\mathcal{M}^N,z)} +$$

$$+ \sum_{k=1}^{g}\left(\sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu}q_k^\tau D^{-1}w_{k\zeta}^\mu U_X^\zeta + \sum_{l=1}^{g} e_l \frac{\partial h^l}{\partial U^\nu}p_k^l D^{-1}w_{k\zeta}^\mu U_X^\zeta\right)$$

(we used the skew-symmetry of the operator $\hat{J}_F^{\mu\xi}$ for the action from the right).

Using again (1.7) and (1.8) we get

$$\hat{\Omega}_{\nu\xi}\hat{J}_F^{\xi\mu} = \hat{I} + \sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu}\left[D^{-1}\frac{d}{dX} - \hat{I}\right]\frac{\partial n^\tau}{\partial U^\xi}g^{\xi\mu}$$

i.e

$$\hat{\Omega}_{\nu\xi}\hat{J}_F^{\xi\mu}f_\mu(X) = f_\mu(X) - \sum_{\tau=1}^{N} \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu}(X)\frac{\partial n^\tau}{\partial U^\xi}(z)g^{\xi\mu}(z)f_\mu(z)$$

for any $f_\mu(X)$. From the definition of $\mathcal{F}_0(z)$ we obtain now the part (II) of the theorem.

Theorem is proved.

Remark. We can say also that the operator $\hat{J}_F^{\nu\xi}\hat{\Omega}_{\xi\mu}$ is identity if acting from the left on $\mathcal{F}_0(z)$ and from the right on $\mathcal{V}_0(z)$. This interpretation will be convenient later for the consideration of the recursion operator.

Let us introduce also the Momentum functional P generating the flow

$$U_T^\nu = U_X^\nu \tag{1.9}$$

with respect to the general bracket (0.6) (with non-degenerate $g^{\nu\mu}(U)$).

Lemma 1.

Any F-bracket (0.6) with non-degenerate tensor $g^{\nu\mu}(U)$ has the local Momentum operator P generating the flow (1.9) on the space $L(\mathcal{M}^N, z)$. The functional P can be written in the form

$$P = \int_{-\infty}^{+\infty} p(U)dX = \frac{1}{2}\int_{-\infty}^{+\infty}\left(\sum_{\tau=1}^N \epsilon^\tau n^\tau n^\tau + \sum_{k=1}^g e_k h^k h^k\right)dX \tag{1.10}$$

where the functions n^τ and h^k correspond to the loop space $L(\mathcal{M}^N, z)$.

Proof.
We should just prove here the relation

$$\frac{\partial p}{\partial U^\nu} = \hat{\Omega}_{\nu\xi}U_X^\xi$$

on $L(\mathcal{M}^N, z)$ according to part (I) of Theorem 1. So we have

$$\sum_{\tau=1}^N \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu}D^{-1}\frac{\partial n^\tau}{\partial U^\xi}U_X^\xi + \sum_{k=1}^g e_k \frac{\partial h^k}{\partial U^\nu}D^{-1}\frac{\partial h^k}{\partial U^\xi}U_X^\xi =$$

$$= \sum_{\tau=1}^N \epsilon^\tau \frac{\partial n^\tau}{\partial U^\nu}n^\tau + \sum_{k=1}^g e_k\frac{\partial h^k}{\partial U^\nu}h^k = \frac{\partial}{\partial U^\nu}p$$

Lemma is proved.

2 The λ-pencils and the integrable hierarchies.

Let us consider now the operator

$$\hat{J}_\lambda^{\nu\mu} = \hat{J}_{(0)}^{\nu\mu} + \lambda\hat{J}_{(1)}^{\nu\mu} = \left(g_{(0)}^{\nu\mu} + \lambda g_{(1)}^{\nu\mu}\right)\frac{d}{dX} + \left(b_{(0)\eta}^{\nu\mu} + \lambda b_{(1)\eta}^{\nu\mu}\right)U_X^\eta +$$

$$+ \sum_{k=1}^{g_0} e_{(0)k}w_{(0)k\eta}^\nu U_X^\eta D^{-1}w_{(0)k\zeta}^\mu U_X^\zeta + \lambda\sum_{k=1}^{g_1} e_{(1)k}w_{(1)k\eta}^\nu U_X^\eta D^{-1}w_{(1)k\zeta}^\mu U_X^\zeta \tag{2.1}$$

for $\hat{J}_{(0)}^{\nu\mu}$ and $\hat{J}_{(1)}^{\nu\mu}$ having the form (0.7). We will call the λ-pencil (2.1) non-degenerate (for small λ) if $\det g_{(0)}^{\nu\mu}(U) \neq 0$.

We will admit here that the linear spaces $\mathcal{W}_{(0)}$ and $\mathcal{W}_{(1)}$ generated by the sets

$$\{w^{\nu}_{(0)1\eta}U^{\eta}_X, \ldots, w^{\nu}_{(0)g_0\eta}U^{\eta}_X\} \text{ and } \{w^{\nu}_{(1)1\eta}U^{\eta}_X, \ldots, w^{\nu}_{(1)g_1\eta}U^{\eta}_X\}$$

can have a nontrivial intersection $\mathcal{V}$.

Let us introduce the basis in $\mathcal{V}$

$$\{\hat{v}^{\nu}_{1\eta}U^{\eta}_X, \ldots, \hat{v}^{\nu}_{d\eta}U^{\eta}_X\}$$

where $d = dim\,\mathcal{V}$ and consider the linear space $\mathcal{W}$ generated by all the flows from $\mathcal{W}_{(0)}$ and $\mathcal{W}_{(1)}$ ($dim\,\mathcal{W} = g_0 + g_1 - d$) with basis

$$\mathcal{B} = \{\hat{w}^{\nu}_{(0)1\eta}U^{\eta}_X, \ldots, \hat{w}^{\nu}_{(0)(g_0-d)\eta}U^{\eta}_X, \hat{v}^{\nu}_{1\eta}U^{\eta}_X, \ldots, \hat{v}^{\nu}_{d\eta}U^{\eta}_X, \hat{w}^{\nu}_{(1)1\eta}U^{\eta}_X, \ldots, \hat{w}^{\nu}_{(1)(g_1-d)\eta}U^{\eta}_X\} =$$

$$= \{\tilde{w}^{\nu}_{1\eta}(U)U^{\eta}_X, \ldots, \tilde{w}^{\nu}_{g_0+g_1-d,\eta}(U)U^{\eta}_X\} \tag{2.2}$$

where $\hat{w}^{\nu}_{(0)k\eta}$ and $\hat{w}^{\nu}_{(1)s\eta}$ are some linear combinations of operators $w_{(0)}$ and $w_{(1)}$ respectively, and the flows corresponding to $\hat{w}_{(0)k}$, $\hat{v}_m$ and $\hat{w}_{(1)s}$ are all linearly independent.

The flows

$$\{\hat{w}^{\nu}_{(0)1\eta}U^{\eta}_X, \ldots, \hat{w}^{\nu}_{(0)(g_0-d)\eta}U^{\eta}_X, \hat{v}^{\nu}_{1\eta}U^{\eta}_X, \ldots, \hat{v}^{\nu}_{d\eta}U^{\eta}_X\}$$

and

$$\{\hat{v}^{\nu}_{1\eta}U^{\eta}_X, \ldots, \hat{v}^{\nu}_{d\eta}U^{\eta}_X, \hat{w}^{\nu}_{(1)1\eta}U^{\eta}_X, \ldots, \hat{w}^{\nu}_{(1)(g_1-d)\eta}U^{\eta}_X\}$$

will give then bases in $\mathcal{W}_{(0)}$ and $\mathcal{W}_{(1)}$ respectively.

The nonlocal part of the bracket $\hat{J}^{\nu\mu}_{\lambda}$

$$\sum_{k=1}^{g_0} e_{(0)k}w^{\nu}_{(0)k\eta}U^{\eta}_X D^{-1}w^{\mu}_{(0)k\zeta}U^{\zeta}_X + \lambda \sum_{s=1}^{g_1} e_{(1)s}w^{\nu}_{(1)s\eta}U^{\eta}_X D^{-1}w^{\mu}_{(1)s\zeta}U^{\zeta}_X$$

will correspond in our case to some quadratic form Q^{ks}_{λ} (linear in λ), $k, s = 1, \ldots, g_0 + g_1 - d$ on the space $\mathcal{W}$.

In our further consideration the question if Q^{ks}_{λ} is non-degenerate on $\mathcal{W}$ for $\lambda \neq 0$ or not will be important and we will mainly consider the pencils (2.1) such that Q^{ks}_{λ} is non-degenerate on $\mathcal{W}$.

Let us formulate now the properties of non-degenerate pencils (2.1) satisfying also the requirement of non-degeneracy of form Q^{ks}_{λ} for $\lambda \neq 0$ connected with the "canonical" integrable hierarchies.

Theorem 2.

Let us consider the non-degenerate pencil (2.1) (det $g^{\nu\mu}_{(0)} \neq 0$) such that the form Q^{ks}_{λ} is also non-degenerate on $\mathcal{W}$ for $\lambda \neq 0$ (small enough). Then:

(I) It is possible to introduce the local functionals:

$$N^\nu(\lambda) = \int_{-\infty}^{+\infty} n^\nu(U,\lambda)dX \ , \quad \nu = 1,\dots,N \tag{2.3}$$

$$P(\lambda) = \int_{-\infty}^{+\infty} p(U,\lambda)dX \tag{2.4}$$

$$H_{(0)}^k(\lambda) = \int_{-\infty}^{+\infty} h_{(0)}^k(U,\lambda)dX \ , \quad k = 1,\dots,g_0 \tag{2.5}$$

and

$$H_{(1)}^s(\lambda) = \int_{-\infty}^{+\infty} h_{(1)}^s(U,\lambda)dX \ , \quad s = 1,\dots,g_1 \tag{2.6}$$

which are the Casimirs, Momentum operator and the Hamiltonian functions for the flows $w_{(0)k\eta}^\nu(U)U_X^\eta$ and $w_{(1)s\eta}^\nu(U)U_X^\eta$ for the bracket $\hat{J}_\lambda^{\nu\mu}$ respectively.

(II) All the functions $n^\nu(U,\lambda)$, $p(U,\lambda)$, $h_{(0)}^k(U,\lambda)$ and $h_{(1)}^s(U,\lambda)$ are regular at $\lambda \to 0$ and can be represented as regular series:

$$n^\nu(U,\lambda) = \sum_{q=0}^{+\infty} n_q^\nu(U)\lambda^q \ , \quad p(U,\lambda) = \sum_{q=0}^{+\infty} p_q(U)\lambda^q \tag{2.7}$$

$$h_{(0)}^k(U,\lambda) = \sum_{q=0}^{+\infty} h_{(0),q}^k(U)\lambda^q \ , \quad h_{(1)}^s(U,\lambda) = \sum_{q=0}^{+\infty} h_{(1),q}^s(U)\lambda^q \tag{2.8}$$

Moreover, we can choose these functionals in such a way that:

$$\frac{\partial p_q}{\partial U^\mu}(z) = 0 \ , \quad \frac{\partial h_{(0),q}^k}{\partial U^\mu}(z) = 0 \ , \quad \frac{\partial h_{(1),q}^s}{\partial U^\mu}(z) = 0 \ , \quad q \geq 0$$

$$\frac{\partial n_q^\nu}{\partial U^\mu}(z) = 0 \ , \quad q \geq 1 \ \text{ and } \ \frac{\partial n_0^\nu}{\partial U^\mu}(z) = e_\mu^\nu$$

where $\det e_\mu^\nu \neq 0$.

(III) The integrals $N^\nu(0)$, $P(0)$, $H_{(0)}^k(0)$ and $H_{(1)}^s(0)$ are the Casimirs, Momentum operator and the Hamiltonian functions for the flows $w_{(0)k\eta}^\nu(U)U_X^\eta$ and $w_{(1)s\eta}^\nu(U)U_X^\eta$ with respect to $\hat{J}_{(0)}^{\nu\mu}$, while the flows generated by the functionals

$$F_q = \int_{-\infty}^{+\infty} f_q(U)dX$$

are connected by the relation:

$$\hat{J}_{(0)}^{\nu\xi} \frac{\partial f_{(q+1)}}{\partial U^\xi} = -\hat{J}_{(1)}^{\nu\xi} \frac{\partial f_q}{\partial U^\xi}$$

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for any functional $F(\lambda)$ from the set (2.3)-(2.6). All the functionals F_q given by the expansions (2.7)-(2.8) generate the local flows and commute with each other with respect to both brackets $\hat{J}_{(0)}$ and $\hat{J}_{(1)}$.

Proof.

Let us put now $\lambda > 0$ and write $\hat{J}^{\nu\mu}_\lambda$ in the form:

$$\hat{J}^{\nu\mu}_\lambda = \left(g^{\nu\mu}_{(0)} + \lambda g^{\nu\mu}_{(1)}\right) \frac{d}{dX} + \left(b^{\nu\mu}_{(0)\eta} + \lambda b^{\nu\mu}_{(1)\eta}\right) U^\eta_X +$$

$$+ \sum_{k=1}^{g_0} e_{(0)k} w^\nu_{(0)k\eta} U^\eta_X D^{-1} w^\mu_{(0)k\zeta} U^\zeta_X + \sum_{k=1}^{g_1} e_{(1)k} \sqrt{\lambda} w^\nu_{(1)k\eta} U^\eta_X D^{-1} \sqrt{\lambda} w^\mu_{(1)k\zeta} U^\zeta_X \qquad (2.9)$$

In the case of the non-degenerate form Q^{ks}_λ on $\mathcal{W}$ we can consider the corresponding embedding of $\mathcal{M}^N$ to $R^{N+g_0+g_1}$ depending on λ. Indeed, all the flows $\tilde{w}_s$ from the set $\mathcal{B}$ will satisfy in this case to conditions

$$g^{\nu\xi}_\lambda \tilde{w}^\mu_{s\xi} = g^{\mu\xi}_\lambda \tilde{w}^\nu_{s\xi} \ , \quad \nabla_\nu \tilde{w}^\mu_{s\eta} = \nabla_\eta \tilde{w}^\mu_{s\nu}$$

$$R^{\nu\mu}_{\eta\zeta} = \sum_{k,s} \left(\tilde{w}^\nu_{k\eta} Q^{ks}_\lambda \tilde{w}^\mu_{s\zeta} - \tilde{w}^\mu_{k\eta} Q^{ks}_\lambda \tilde{w}^\nu_{s\zeta}\right)$$

for non-degenerate $g^{\mu\xi}_\lambda$ according to Ferapontov theorem.

Since the operators $w^\nu_{(0)k\eta}(U)$, $w^\nu_{(1)s\eta}(U)$ are just linear combinations of $\tilde{w}^\nu_{n\eta}(U)$ (with constant coefficients) and the form Q^{ks}_λ coincides with the nonlocal part of (2.9) we will have the corresponding Gauss and Petersson-Codazzi equations for these flows and the curvature tensor $R^{\nu\mu}_{\eta\zeta}$. So for non-degenerate $g^{\nu\mu}_\lambda$ we will get the local embedding of $\mathcal{M}^N$ to $R^{N+g_0+g_1}$ (actually to some subspace $R^{N+g_0+g_1-d} \subset R^{N+g_0+g_1}$) depending on λ.

Since the embedding is defined up to the Poincare transformation in $R^{N+g_0+g_1}$ we can choose at all λ the coordinates Z^I, $I = 1, \dots, N + g_0 + g_1$ in such a way that:

1) All $Z^I = 0$ at the point $z \in \mathcal{M}^N$ and the metric G^{IJ} in $R^{N+g_0+g_1}$ has the form

$$G^{IJ} = E^I \delta^{IJ}$$

where $E^{N+k} = e_{(0)k}$, $k = 1, \dots, g_0$, $E^{N+g_0+s} = e_{(1)s}$, $s = 1, \dots, g_1$.

2) The first N coordinates Z^ν, $\nu = 1, \dots, N$ are tangential to $\mathcal{M}^N$ at the point $z \in \mathcal{M}^N$ at all λ.

3) The last $g_0 + g_1$ coordinates are orthogonal to $\mathcal{M}^N$ at the point z and the Weingarten operators

$$w^\nu_{(0)1\eta}(U), \dots, w^\nu_{(0)g_0\eta}(U), \sqrt{\lambda} w^\nu_{(1)1\eta}(U), \dots, \sqrt{\lambda} w^\nu_{(1)g_1\eta}(U)$$

correspond to the parallel vector fields $\mathbf{N}_{(0)(k)}(U)$, $\mathbf{N}_{(1)(s)}(U)$ in the normal bundle such that:

$$\mathbf{N}_{(0)(k)}(z) = \left(0, \dots, 0, -E^{N+k}, 0, \dots, 0\right)^T$$

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$(-E^{N+k}$ stays at the position $N+k)$ $k = 1, \ldots, g_0$,

$$\mathbf{N}_{(1)(k)}(z) = \left(0, \ldots, 0, -E^{N+g_0+s}, 0, \ldots, 0\right)^T$$

$(-E^{N+g_0+s}$ stays at the position $N+g_0+s)$ $s = 1, \ldots, g_1$.

So, according to [19] the restriction of the first N coordinates $Z^1, \ldots, Z^N$ gives the Casimirs of the bracket (2.9) on $L(\mathcal{M}^N, z)$

$$\tilde{N}^\nu(\lambda) = \int_{-\infty}^{+\infty} \tilde{n}^\nu(U, \lambda)dX = \int_{-\infty}^{+\infty} Z^\nu|_{\mathcal{M}^N}(U, \lambda)dX \ , \quad \nu = 1, \ldots, N \tag{2.10}$$

while the restrictions of the last $g_0 + g_1$ coordinates give the Hamiltonian functions for the flows $w^\nu_{(0)k\eta}U^\eta_X$

$$\tilde{H}^k_{(0)}(\lambda) = \int_{-\infty}^{+\infty} \tilde{h}^k_{(0)}(U, \lambda)dX = \int_{-\infty}^{+\infty} Z^{N+k}|_{\mathcal{M}^N}(U, \lambda)dX \ , \quad k = 1, \ldots, g_0 \tag{2.11}$$

and $\sqrt{\lambda}w^\nu_{(1)s\eta}U^\eta_X$

$$\tilde{H}^s_{(1)}(\lambda) = \int_{-\infty}^{+\infty} \tilde{h}^s_{(1)}(U, \lambda)dX = \int_{-\infty}^{+\infty} Z^{N+g_0+s}|_{\mathcal{M}^N}(U, \lambda)dX \ , \quad s = 1, \ldots, g_1 \tag{2.12}$$

Remark. For $\lambda < 0$ the signature of $R^{N+g_0+g_1}$ may be different from the case $\lambda > 0$ but all the statements of the Theorem will certainly be also true.

Let us study now the λ-dependence of the functions $\tilde{n}^\nu(U, \lambda)$, $\tilde{h}^k_{(0)}(U, \lambda)$ and $\tilde{h}^s_{(1)}(U, \lambda)$ at $\lambda \to 0$.

Consider N tangential vectors to $\mathcal{M}^N$ corresponding to the coordinate system $\{U^\nu\}$, i.e.

$$\mathbf{e}_{(\nu)} = \left(\frac{\partial Z^1}{\partial U^\nu}, \ldots, \frac{\partial Z^{N+g_0+g_1}}{\partial U^\nu}\right)^T \ , \quad \nu = 1, \ldots, N$$

We have the following relations (in $R^{N+g_0+g_1}$) for the differentials of $\mathbf{e}_{(\nu)}(U, \lambda)$, $\mathbf{N}_{(0)(k)}(U, \lambda)$ and $\mathbf{N}_{(1)(s)}(U, \lambda)$ on $\mathcal{M}^N$:

$$d\mathbf{e}_{(\nu)} = \Gamma^\mu_{\nu\eta}(U, \lambda)\mathbf{e}_{(\mu)}dU^\eta -$$

$$-\sum_{k=1}^{g_0} E^{N+k}g_{\nu\mu}(U, \lambda)w^\mu_{(0)k\eta}(U, \lambda)\mathbf{N}_{(0)(k)}dU^\eta - \sqrt{\lambda}\sum_{s=1}^{g_1} E^{N+g_0+s}g_{\nu\mu}(U, \lambda)w^\mu_{(1)s\eta}(U, \lambda)\mathbf{N}_{(1)(s)}dU^\eta$$

$$d\mathbf{N}_{(0)(k)} = w^\nu_{(0)k\eta}(U, \lambda)\mathbf{e}_{(\nu)}dU^\eta$$

$$d\mathbf{N}_{(1)(s)} = \sqrt{\lambda}w^\nu_{(1)s\eta}(U, \lambda)\mathbf{e}_{(\nu)}dU^\eta$$

where $\Gamma^\mu_{\nu\eta} = -g_{\nu\xi}b^{\xi\mu}_\eta$, $g_{\nu\xi}g^{\xi\mu} \equiv \delta^\mu_\nu$.

So for any curve $\gamma(t)$ on $\mathcal{M}^N$ ($\gamma(0) = z$) we have the evolution system for $\mathbf{e}_{(\nu)}(t)$, $\mathbf{N}_{(0)(k)}(t)$ and $\mathbf{N}_{(1)(s)}(t)$ having the general form:

$$\frac{d}{dt} \begin{pmatrix} \mathbf{e}_{(\nu)}(t) \\ \mathbf{N}_{(0)(k)}(t) \\ \mathbf{N}_{(1)(s)}(t) \end{pmatrix} = \begin{pmatrix} *(t,\lambda) & *(t,\lambda) & \sqrt{\lambda}\,*(t,\lambda) \\ *(t,\lambda) & 0 & 0 \\ \sqrt{\lambda}\,*(t,\lambda) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_{(\nu)}(t) \\ \mathbf{N}_{(0)(k)}(t) \\ \mathbf{N}_{(1)(s)}(t) \end{pmatrix} \qquad (2.13)$$

where all $*(t,\lambda)$ are regular at $\lambda \to 0$ (matrix-) functions of λ.

The formal solution of (2.13) can be written as the chronological exponent

$$T \exp\left(\int_0^t \hat{H}(t) dt \right) = \hat{I} + \sum_{n=1}^{+\infty} \frac{1}{n!} \int_0^{t_1} \ldots \int_0^{t_n} T\left(\hat{H}(t_1) \ldots \hat{H}(t_n) \right) dt_1 \ldots dt_n$$

applied to the initial data $\mathbf{e}_{(\nu)}(0)$, $\mathbf{N}_{(0)(k)}(0)$, $\mathbf{N}_{(1)(s)}(0)$, where $\hat{H}(t)$ is the matrix of the system (2.13).

It is easy to verify now that for any $n \geq 1$ we have

$$T\left(\hat{H}(t_1) \ldots \hat{H}(t_n) \right) = \begin{pmatrix} *(t,\lambda) & *(t,\lambda) & \sqrt{\lambda}\,*(t,\lambda) \\ *(t,\lambda) & *(t,\lambda) & \sqrt{\lambda}\,*(t,\lambda) \\ \sqrt{\lambda}\,*(t,\lambda) & \sqrt{\lambda}\,*(t,\lambda) & \lambda\,*(t,\lambda) \end{pmatrix}$$

where all $*(t,\lambda)$ are regular matrix-functions at $\lambda \to 0$.

For the densities of Casimirs (2.10) and Hamiltonian functions (2.11)-(2.12) we can write now the equations

$$\frac{d\tilde{n}^\nu(t,\lambda)}{dt} = \left[U_t^\mu \mathbf{e}_{(\mu)}(t,\lambda) \right]^\nu = \epsilon^\nu \langle U_t^\mu \mathbf{e}_{(\mu)}(t,\lambda), \mathbf{e}_\nu(0) \rangle$$

$$\frac{d\tilde{h}_{(0)}^k(t,\lambda)}{dt} = -\langle U_t^\mu \mathbf{e}_{(\mu)}(t,\lambda), \mathbf{N}_{(0)(k)}(0) \rangle$$

$$\frac{d\tilde{h}_{(1)}^s(t,\lambda)}{dt} = -\langle U_t^\mu \mathbf{e}_{(\mu)}(t,\lambda), \mathbf{N}_{(1)(s)}(0) \rangle$$

along the same curve $\gamma(t)$ on $\mathcal{M}^N$. Since $\gamma(t)$ is just the arbitrary curve we have that

$$\tilde{n}^\nu(U,\lambda) = *(U,\lambda) \ , \ \ \tilde{h}_{(0)}^k(U,\lambda) = *(U,\lambda) \ , \ \ \tilde{h}_{(1)}^s(U,\lambda) = \sqrt{\lambda}\,*(U,\lambda)$$

where $*(U,\lambda)$ are regular at $\lambda \to 0$.

It is easy to see that the expression (1.10) for the Momentum Operator is regular at $\lambda \to 0$ in this case and we can put

$$h_{(0)}^k(U,\lambda) = \tilde{h}_{(0)}^k(U,\lambda) \ , \ \ h_{(1)}^s(U,\lambda) = \frac{1}{\sqrt{\lambda}} \tilde{h}_{(1)}^s(U,\lambda)$$

to be the regular at $\lambda \to 0$ densities of Hamiltonian functions $H_{(0)}^k(\lambda)$, $H_{(1)}^s(\lambda)$.

According to the geometric construction we have

$$\frac{\partial h_{(0)}^k(U,\lambda)}{\partial U^\mu}\Big|_z \equiv 0 \ , \ \ \frac{\partial h_{(1)}^s(U,\lambda)}{\partial U^\mu}\Big|_z \equiv 0$$

and also

$$\frac{\partial p(U,\lambda)}{\partial U^\mu}\Big|_z \equiv 0$$

Since the functions $\tilde{n}^\nu(U,\lambda)$ give locally the coordinate system on $\mathcal{M}^N$ at every λ we have

$$det\left(\frac{\partial \tilde{n}^\nu}{\partial U^\mu}(z,\lambda)\right) \neq 0$$

and we can put

$$n^\nu(U,\lambda) = \frac{\partial \tilde{n}^\nu}{\partial U^\xi}(z,0)\,\frac{\partial U^\xi}{\partial \tilde{n}^\eta}(z,\lambda)\,\tilde{n}^\eta(U,\lambda)$$

such that

$$\frac{\partial n^\nu(U,\lambda)}{\partial U^\mu}\Big|_z \equiv \frac{\partial n^\nu}{\partial U^\mu}(z,0) = (\mathbf{e}_{(\mu)})^\nu \ , \ \ \nu = 1,\ldots,N$$

where $\mathbf{e}_{(\mu)}$ are the tangent vectors at the point z for $\lambda = 0$, $n^\nu(U,0) = \tilde{n}^\nu(U,0)$. So we get parts (I) and (II) of the theorem.

For the part (III) we prove here first the Lemma:

Lemma 2.
Under the conditions formulated in the theorem the functionals

$$N_p^\nu = \int_{-\infty}^{+\infty} n_p^\nu(U)dX \ , \ \ P_q = \int_{-\infty}^{+\infty} p_q(U)dX \tag{2.14}$$

$$H_{(0)l}^k = \int_{-\infty}^{+\infty} h_{(0)l}^k(U)dX \ , \ \ H_{(1)t}^s = \int_{-\infty}^{+\infty} h_{(1)t}^s(U)dX \tag{2.15}$$

$p,q,l,t = 0,1,\ldots$, generate the local flows with respect to both $\hat{J}_{(0)}$ and $\hat{J}_{(1)}$ and commute with the functionals N_0^μ, P_0, $H_{(0)0}^m$ and $H_{(1)0}^n$ with respect to the bracket $\hat{J}_{(0)}$

Proof.
Since the functionals N_0^μ are the annihilators of $\hat{J}_{(0)}$ they commute with all other functionals with respect to $\hat{J}_{(0)}$ on $L(\mathcal{M}^N, z)$. Also from the translational invariance of all functionals N_p^ν, P_q, $H_{(0)l}^k$, $H_{(1)t}^s$ we get that they commute also with the momentum operator P_0 of $\hat{J}_{(0)}$ with respect to $\hat{J}_{(0)}$.

We know then that the functionals $N^\nu(\lambda)$, $P(\lambda)$, $H_{(0)}^k(\lambda)$ and $H_{(1)}^s(\lambda)$ generate the local flows with respect to $\hat{J}_\lambda$. In the case of non-degenerate form Q_λ^{ks} (on $\mathcal{W}$) this means that they are the conservation laws for all the flows $\tilde{w}_{n\eta}^\nu(U)U_X^\eta$ introduced in (2.2). So they are the conservation laws for the flows $w_{(0)k\eta}^\nu(U)U_X^\eta$ and $w_{(1)s\eta}^\nu(U)U_X^\eta$ and generate the local flows with respect to

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$\hat{J}_{(0)}$ and $\hat{J}_{(1)}$. Now since the flows $w^\nu_{(0)m\eta}(U)U^\eta_X$, $w^\nu_{(1)n\eta}(U)U^\eta_X$ are generated by the functionals $H^m_{(0)0}$ and $H^n_{(1)0}$ with respect to $\hat{J}_{(0)}$ we get that all N^ν_p, P_q, $H^k_{(0)l}$, $H^s_{(1)t}$ should commute with $H^m_{(0)0}$ and $H^n_{(1)0}$ with respect to $\hat{J}_{(0)}$ on $L(\mathcal{M}^N, z)$.

<div align="right">Lemma is proved.</div>

For the commutativity of all N^ν_p, P_q, $H^k_{(0)l}$, $H^s_{(1)t}$ with respect to both brackets we can now use just the common approach for the bi-Hamiltonian systems ([17]) writing

$$\delta F_q \hat{J}_{(0)} \delta G_k = -\delta F_{q-1} \hat{J}_{(1)} \delta G_k = \delta F_{q-1} \hat{J}_{(0)} \delta G_{k+1} = \ldots = \delta F_0 \hat{J}_{(0)} \delta G_{k+q} = 0$$

for any two functionals F_q and G_k from the set (2.14)-(2.15) (the same for the bracket $\hat{J}_{(1)}$).

<div align="right">Theorem is proved.</div>

Remark.

Let us point out here that the requirement of non-degeneracy of the form Q^{ks}_λ on $\mathcal{W}$ is important and in general Theorem 2 is not true without it. As the example we consider here the Poisson pencil $\hat{J}_\lambda = \hat{J}_{(0)} + \lambda \hat{J}_{(1)}$ where

$$\hat{J}_{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dX} + \begin{pmatrix} U^1 & U^2 \\ -U^2 & -U^1 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} D^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} +$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} D^{-1} \begin{pmatrix} U^1 & U^2 \\ -U^2 & -U^1 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} \qquad (2.16)$$

and

$$\hat{J}_{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dX} + \begin{pmatrix} 0 & 0 \\ 2U^1 & 0 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} D^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} +$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} D^{-1} \begin{pmatrix} 0 & 0 \\ 2U^1 & 0 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} \qquad (2.17)$$

We have here the three-dimensional space $\mathcal{W}$ generated by the flows

$$\tilde{w}^\nu_{1\eta} U^\eta_X = \begin{pmatrix} U^1 & U^2 \\ -U^2 & -U^1 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} , \quad \tilde{w}^\nu_{2\eta} U^\eta_X = \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} , \quad \tilde{w}^\nu_{3\eta} U^\eta_X = \begin{pmatrix} 0 & 0 \\ 2U^1 & 0 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix}$$

Operator $\hat{J}_\lambda$ can be written as

$$\hat{J}_\lambda = \begin{pmatrix} 1 & \lambda \\ \lambda & -1 \end{pmatrix} \frac{d}{dX} + \begin{pmatrix} U^1 & U^2 \\ -U^2 + 2\lambda U^1 & -U^1 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} D^{-1} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} +$$

$$+ \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix} D^{-1} \begin{pmatrix} U^1 & U^2 \\ -U^2 + 2\lambda U^1 & -U^1 \end{pmatrix} \begin{pmatrix} U^1_X \\ U^2_X \end{pmatrix}$$

and the form

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$$Q_\lambda = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix}$$

is degenerate on $\mathcal{W}$.

The metric

$$g_\lambda^{\nu\mu}(U) = \begin{pmatrix} 1 & \lambda \\ \lambda & -1 \end{pmatrix}$$

here is just the flat metric on R^2, $det\, g_\lambda^{\nu\mu} \neq 0$, and it is easy to check the equations

$$g_\lambda^{\nu\xi} W_{1\xi}^\mu(\lambda) = g_\lambda^{\mu\xi} W_{1\xi}^\nu(\lambda) \ , \ \ g_\lambda^{\nu\xi} W_{2\xi}^\mu = g_\lambda^{\mu\xi} W_{2\xi}^\nu$$

where

$$W_{1\xi}^\mu(\lambda) = \begin{pmatrix} U^1 & U^2 \\ -U^2 + 2\lambda U^1 & -U^1 \end{pmatrix} \ , \ \ W_{2\xi}^\mu(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Also

$$\partial_\nu W_{1\mu}^\xi(\lambda) = \partial_\mu W_{1\nu}^\xi(\lambda) \ , \ \ \partial_\nu W_{2\mu}^\xi = \partial_\mu W_{2\nu}^\xi$$

in the flat coordinates U^1, U^2 and

$$R_{12}^{12} = W_{11}^1(\lambda) W_{22}^2 + W_{21}^1 W_{12}^2(\lambda) - W_{11}^2(\lambda) W_{22}^1 - W_{21}^2 W_{12}^1(\lambda) =$$

$$= W_{11}^1(\lambda) + W_{12}^2(\lambda) \equiv 0$$

so $\hat{J}_\lambda$ represents a Poisson bracket of Ferapontov type for all λ.

Both $g_{(0)}^{\nu\mu}$ and $g_{(1)}^{\nu\mu}$ are non-degenerate in this case. However, the flow $\tilde{w}_{1\eta}^\nu U_X^\eta$ is not Hamiltonian with respect to $\hat{J}_{(1)}$ and $\tilde{w}_{3\eta}^\nu U_X^\eta$ is not Hamiltonian with respect to $\hat{J}_{(0)}$ as follows from

$$g_{(0)}^{\nu\xi} \tilde{w}_{3\xi}^\mu \neq g_{(0)}^{\mu\xi} \tilde{w}_{3\xi}^\nu \ , \ \ g_{(1)}^{\nu\xi} \tilde{w}_{1\xi}^\mu \neq g_{(1)}^{\mu\xi} \tilde{w}_{1\xi}^\nu$$

It is also easy to check that the flows $\tilde{w}_{1\eta}^\nu U_X^\eta$ and $\tilde{w}_{3\eta}^\nu U_X^\eta$ do not commute with each other.

3 The Recursion Operator and the Higher Hamiltonian structures.

Let us use now the symplectic operator (1.1) for non-degenerate bracket $\hat{J}_{(0)}$ and consider the Recursion Operator $\hat{R}_\mu^\nu = \hat{J}_{(1)}^{\nu\tau} \hat{\Omega}_{(0)\tau\mu}$ under the assumptions of Theorem 2.

We put $v^\tau(U, \lambda) = n^\tau(U, \lambda)$, $\tau = 1, \ldots, N$, $v^{N+k}(U, \lambda) = h_{(0)}^k(U, \lambda)$, $k = 1, \ldots, g_0$, $v_0^s(U) \equiv v^s(U, 0)$, $E_{(0)}^\tau = \epsilon^\tau$, $\tau = 1, \ldots, N$, $E_{(0)}^{N+k} = e_{(0)k}$, $k = 1, \ldots, g_0$ (where $n^\tau(U, \lambda)$, $h_{(0)}^k(U, \lambda)$ are the functions from Theorem 2) and write the symplectic form $\hat{\Omega}_{(0)\tau\mu}$ as

$$\hat{\Omega}_{(0)\tau\mu} = \sum_{k=1}^{N+g_0} E_{(0)}^k \frac{\partial v_0^k}{\partial U^\tau} D^{-1} \frac{\partial v_0^k}{\partial U^\mu} \tag{3.1}$$

We can write also the operator $\hat{J}_{(0)}^{\nu\mu}$ as

$$\hat{J}_{(0)}^{\nu\mu} = g_{(0)}^{\nu\mu} \frac{d}{dX} + b_{(0)\eta}^{\nu\mu} U_X^\eta + \sum_{k=1}^{N+g_0} E_{(0)}^k \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial v_0^k}{\partial U^\tau} \right] D^{-1} \left[\hat{J}_{(0)}^{\mu\sigma} \frac{\partial v_0^k}{\partial U^\sigma} \right] \tag{3.2}$$

For $\hat{R}_\mu^\nu$ we have the expression:

$$\hat{R}_\mu^\nu = \hat{J}_{(1)}^{\nu\tau} \hat{\Omega}_{(0)\tau\mu} = \sum_{k=1}^{N+g_0} E_{(0)}^k g_{(1)}^{\nu\tau} \left(\frac{\partial v_0^k}{\partial U^\tau} \right)_X D^{-1} \frac{\partial v_0^k}{\partial U^\mu} + \sum_{k=1}^{N+g_0} E_{(0)}^k g_{(1)}^{\nu\tau} \frac{\partial v_0^k}{\partial U^\tau} \frac{\partial v_0^k}{\partial U^\mu} +$$

$$+ \sum_{k=1}^{N+g_0} E_{(0)}^k b_{(1)\eta}^{\nu\tau} U_X^\eta \frac{\partial v_0^k}{\partial U^\tau} D^{-1} \frac{\partial v_0^k}{\partial U^\mu} + \sum_{s=1}^{g_1} \sum_{k=1}^{N+g_0} e_{(1)s} E_{(0)}^k w_{(1)s\eta}^\nu U_X^\eta D^{-1} w_{(1)s\zeta}^\tau U_X^\zeta \frac{\partial v_0^k}{\partial U^\tau} D^{-1} \frac{\partial v_0^k}{\partial U^\mu}$$

where

$$w_{(1)s\zeta}^\tau U_X^\zeta \frac{\partial v_0^k}{\partial U^\tau} \equiv \left(Q_s^k \right)_X = \frac{d}{dX} Q_s^k - Q_s^k \frac{d}{dX}$$

for some $Q_s^k(U)$ according to Theorem 2.

Let us normalize the functions $Q_s^k(U)$ such that $Q_s^k(z) = 0$ and we have

$$\ldots D^{-1} \frac{d}{dX} Q_s^k \ldots = \ldots Q_s^k \ldots$$

on $L(\mathcal{M}^N, z)$.

Also $d/dX \, D^{-1} \equiv \hat{I}$ on $L(\mathcal{M}^N, z)$ and

$$\sum_{k=1}^{N+g_0} E_{(0)}^k g_{(1)}^{\nu\tau} \left(\frac{\partial v_0^k}{\partial U^\tau} \right)_X D^{-1} \frac{\partial v_0^k}{\partial U^\mu} + \sum_{k=1}^{N+g_0} E_{(0)}^k b_{(1)\eta}^{\nu\tau} U_X^\eta \frac{\partial v_0^k}{\partial U^\tau} D^{-1} \frac{\partial v_0^k}{\partial U^\mu} +$$

$$+ \sum_{s=1}^{g_1} \sum_{k=1}^{N+g_0} e_{(1)s} E_{(0)}^k w_{(1)s\eta}^\nu U_X^\eta Q_s^k D^{-1} \frac{\partial v_0^k}{\partial U^\mu} = \sum_{k=1}^{N+g_0} E_{(0)}^k \left[\hat{J}_{(1)}^{\nu\tau} \frac{\partial v_0^k}{\partial U^\tau} \right] D^{-1} \frac{\partial v_0^k}{\partial U^\mu}$$

Using also the relation

$$\sum_{k=1}^{N+g_0} E_{(0)}^k \frac{\partial v_0^k}{\partial U^\mu} Q_s^k = \hat{\Omega}_{(0)\mu\tau} w_{(1)s\eta}^\tau U_X^\eta = \frac{\partial h_{(1)}^s(U, 0)}{\partial U^\mu} = \frac{\partial h_{(1)0}^s(U)}{\partial U^\mu}$$

on $L(\mathcal{M}^N, z)$ (since $\partial h_{(1)q}^s / \partial U^\mu (z) = 0$, $q \geq 0$) we can write

$$\hat{R}_\mu^\nu = g_{(1)}^{\nu\tau} g_{(0)\tau\mu} + \sum_{k=1}^{N+g_0} E_{(0)}^k \left[\hat{J}_{(1)}^{\nu\tau} \frac{\partial v_0^k}{\partial U^\tau} \right] D^{-1} \frac{\partial v_0^k}{\partial U^\mu} - \sum_{s=1}^{g_1} e_{(1)s} w_{(1)s\eta}^\nu U_X^\eta D^{-1} \frac{\partial h_{(1)0}^s}{\partial U^\mu} =$$

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$$= V_\mu^\nu - \sum_{k=1}^{N+g_0} E_{(0)}^k \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial v_1^k}{\partial U^\tau} \right] D^{-1} \frac{\partial v_0^k}{\partial U^\mu} - \sum_{s=1}^{g_1} e_{(1)s} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial h_{(1)0}^s}{\partial U^\tau} \right] D^{-1} \frac{\partial h_{(1)0}^s}{\partial U^\mu}$$

where

$$v^k(U, \lambda) = \sum_{q=0}^{+\infty} v_q^k(U)\lambda^q \;,\;\; h_{(1)}^s(U, \lambda) = \sum_{q=0}^{+\infty} h_{(1)q}^s(U)\lambda^q$$

and $V_\mu^\nu(U) = g_{(1)}^{\nu\tau}(U) g_{(0)\tau\mu}(U)$.

Let us mention here that according to this definition $\hat{R}_\mu^\nu$ will act from the left on the vector-fields (on $\mathcal{V}_0(z)$) and from the right on the gradients of functionals on $L(\mathcal{M}^N, z)$.

Theorem 3.

Let us consider the non-degenerate pencil (2.1) with $\det g_{(0)}^{\nu\mu} \neq 0$ such that the form Q_λ^{ks} is also non-degenerate on $\mathcal{W}$ for small enough $\lambda \neq 0$. Then:

(I) Any power $[\hat{R}^n]$, $n \geq 1$ of the recursion operator can be written in the form:

$$\left[\hat{R}^n \right]_\mu^\nu = \left[\hat{V}^n \right]_\mu^\nu + (-1)^n \sum_{k=1}^{N+g_0} E_{(0)}^k \left(\sum_{s=1}^{n} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial v_s^k}{\partial U^\tau} \right] D^{-1} \frac{\partial v_{n-s}^k}{\partial U^\mu} \right) +$$

$$+ (-1)^n \sum_{k=1}^{g_1} e_{(1)k} \left(\sum_{s=1}^{n} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial h_{(1),s-1}^k}{\partial U^\tau} \right] D^{-1} \frac{\partial h_{(1),n-s}^k}{\partial U^\mu} \right) \tag{3.3}$$

(II) The higher Hamiltonian structures $\hat{J}_{(n)}^{\nu\mu} = [\hat{R}^n]_\xi^\nu \hat{J}_{(0)}^{\xi\mu}$ can be written on $\mathcal{F}_0(z)$ in the following weakly-nonlocal form:

$$\hat{J}_{(n)}^{\nu\mu} = \left[\hat{V}^n \right]_\xi^\nu g_{(0)}^{\xi\mu} \frac{d}{dX} + \left[\hat{V}^n \right]_\xi^\nu b_{(0)\eta}^{\xi\mu} U_X^\eta +$$

$$+ (-1)^n \sum_{k=1}^{N+g_0} E_{(0)}^k \left(\sum_{s=1}^{n} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial v_s^k}{\partial U^\tau} \right] \frac{\partial v_{n-s}^k}{\partial U^\xi} g_{(0)}^{\xi\mu} \right) +$$

$$+ (-1)^n \sum_{k=1}^{g_1} e_{(1)k} \left(\sum_{s=1}^{n} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial h_{(1),s-1}^k}{\partial U^\tau} \right] \frac{\partial h_{(1),n-s}^k}{\partial U^\xi} g_{(0)}^{\xi\mu} \right) +$$

$$+ (-1)^{n+1} \sum_{k=1}^{N+g_0} E_{(0)}^k \left(\sum_{s=1}^{n-1} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial v_s^k}{\partial U^\tau} \right] D^{-1} \left[\hat{J}_{(0)}^{\mu\xi} \frac{\partial v_{n-s}^k}{\partial U^\xi} \right] \right) +$$

$$+ (-1)^{n+1} \sum_{k=1}^{g_1} e_{(1)k} \left(\sum_{s=1}^{n} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial h_{(1),s-1}^k}{\partial U^\tau} \right] D^{-1} \left[\hat{J}_{(0)}^{\mu\xi} \frac{\partial h_{(1),n-s}^k}{\partial U^\xi} \right] \right) \tag{3.4}$$

for $n \geq 2$.

(III) All the "negative" symplectic forms

$$\hat{\Omega}_{(-n)\nu\mu} = \hat{\Omega}_{(0)\nu\mu}\left[\hat{R}^n\right]_\mu^\xi \ , \ \ n \geq 1$$

can be represented on $\mathcal{V}_0(z)$ in the form:

$$\hat{\Omega}_{(-n)\nu\mu} = (-1)^n \sum_{k=1}^{N+g_0} E_{(0)}^k \left(\sum_{s=0}^n \frac{\partial v_s^k}{\partial U^\nu} D^{-1} \frac{\partial v_{n-s}^k}{\partial U^\mu} \right) + (-1)^n \sum_{k=1}^{g_1} e_{(1)k} \left(\sum_{s=1}^n \frac{\partial h_{(1),s-1}^k}{\partial U^\nu} D^{-1} \frac{\partial h_{(1),n-s}^k}{\partial U^\mu} \right)$$

$$(3.5)$$

Proof.

(I) We have by induction:

$$\hat{R}_\xi^\nu \left[\hat{R}^n\right]_\mu^\xi = \left[\hat{V}^{n+1}\right]_\mu^\nu + (-1)^n \sum_{k=1}^{N+g_0} E_{(0)}^k \left(\sum_{s=1}^n \left[\hat{R}_\xi^\nu \hat{J}_{(0)}^{\xi\sigma} \frac{\partial v_s^k}{\partial U^\sigma} \right] D^{-1} \frac{\partial v_{n-s}^k}{\partial U^\mu} \right) +$$

$$+ (-1)^n \sum_{k=1}^{g_1} e_{(1)k} \left(\sum_{s=1}^n \left[\hat{R}_\xi^\nu \hat{J}_{(0)}^{\xi\sigma} \frac{\partial h_{(1),s-1}^k}{\partial U^\sigma} \right] D^{-1} \frac{\partial h_{(1),n-s}^k}{\partial U^\mu} \right) +$$

$$+ (-1)^n \sum_{q=1}^{N+g_0} \sum_{k=1}^{N+g_0} E_{(0)}^q \left(\sum_{s=1}^n \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial v_1^q}{\partial U^\tau} \right] D^{-1} P_{0s}^{qk} \frac{\partial v_{n-s}^k}{\partial U^\mu} \right) +$$

$$+ (-1)^n \sum_{q=1}^{N+g_0} \sum_{k=1}^{g_1} E_{(0)}^q \left(\sum_{s=1}^n \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial v_1^q}{\partial U^\tau} \right] D^{-1} Q_{0s}^{qk} \frac{\partial h_{(1),n-s}^k}{\partial U^\mu} \right) -$$

$$- \sum_{q=1}^{N+g_0} E_{(0)}^q \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial v_1^q}{\partial U^\tau} \right] D^{-1} \frac{\partial v_0^q}{\partial U^\xi} \left[\hat{V}^n\right]_\mu^\xi +$$

$$+ (-1)^n \sum_{q=1}^{g_1} \sum_{k=1}^{N+g_0} e_{(1)q} \left(\sum_{s=1}^n \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial h_{(1),0}^q}{\partial U^\tau} \right] D^{-1} S_{0s}^{qk} \frac{\partial v_{n-s}^k}{\partial U^\mu} \right) +$$

$$+ (-1)^n \sum_{q=1}^{g_1} \sum_{k=1}^{g_1} e_{(1)q} \left(\sum_{s=1}^n \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial h_{(1),0}^q}{\partial U^\tau} \right] D^{-1} T_{0s}^{qk} \frac{\partial h_{(1),n-s}^k}{\partial U^\mu} \right) -$$

$$- \sum_{q=1}^{g_1} e_{(1)q} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial h_{(1),0}^q}{\partial U^\tau} \right] D^{-1} \frac{\partial h_{(1),0}^q}{\partial U^\xi} \left[\hat{V}^n\right]_\mu^\xi$$

where we have

$$E_{(0)}^k \frac{\partial v_0^q}{\partial U^\xi} \left[\hat{J}_{(0)}^{\xi\sigma} \frac{\partial v_s^k}{\partial U^\sigma} \right] \equiv \left(P_{0s}^{qk} \right)_X = \frac{d}{dX} P_{0s}^{qk} - P_{0s}^{qk} \frac{d}{dX}$$

$$e_{(1)k} \frac{\partial v_0^q}{\partial U^\xi} \left[\hat{J}_{(0)}^{\xi\sigma} \frac{\partial h_{(1),s}^k}{\partial U^\sigma} \right] \equiv \left(Q_{0s}^{qk} \right)_X = \frac{d}{dX} Q_{0s}^{qk} - Q_{0s}^{qk} \frac{d}{dX}$$

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$$E_{(0)}^k \frac{\partial h_{(1),0}^q}{\partial U^\xi}\left[\hat{J}_{(0)}^{\xi\sigma}\frac{\partial v_s^k}{\partial U^\sigma}\right] \equiv \left(S_{0s}^{qk}\right)_X = \frac{d}{dX}S_{0s}^{qk} - S_{0s}^{qk}\frac{d}{dX}$$

$$e_{(1)k}\frac{\partial h_{(1),0}^q}{\partial U^\xi}\left[\hat{J}_{(0)}^{\xi\sigma}\frac{\partial h_{(1),s}^k}{\partial U^\sigma}\right] \equiv \left(T_{0s}^{qk}\right)_X = \frac{d}{dX}T_{0s}^{qk} - T_{0s}^{qk}\frac{d}{dX}$$

for some functions $P_{0s}^{qk}(U)$, $Q_{0s}^{qk}(U)$, $S_{0s}^{qk}(U)$, $T_{0s}^{qk}(U)$ according to Theorem 2 and we use the normalization:

$$P_{0s}^{qk}(z) = Q_{0s}^{qk}(z) = S_{0s}^{qk}(z) = T_{0s}^{qk}(z) = 0$$

We have from Theorem 2:

$$\frac{\partial v_s^k}{\partial U^\tau}(z) = 0 \;, \;\; s \geq 1 \;\; \left(\text{actually } \frac{\partial h_{(0),0}^k}{\partial U^\tau}(z) = 0 \;\; \text{also}\right)$$

$$\frac{\partial h_{(1),s-1}^k}{\partial U^\tau}(z) = 0 \;, \;\; s \geq 1$$

so for $s \geq 1$ we can write

$$\left[\hat{\Omega}_{(0)\nu\xi}\hat{J}_{(0)}^{\xi\tau}\frac{\partial v_s^k}{\partial U^\tau}\right] = \frac{\partial v_s^k}{\partial U^\nu} \;, \;\; \left[\hat{\Omega}_{(0)\nu\xi}\hat{J}_{(0)}^{\xi\tau}\frac{\partial h_{(1),s-1}^k}{\partial U^\tau}\right] = \frac{\partial h_{(1),s-1}^k}{\partial U^\nu}$$

on $L(\mathcal{M}^N, z)$ and

$$\left[\hat{R}_\xi^\nu\hat{J}_{(0)}^{\xi\tau}\frac{\partial v_s^k}{\partial U^\tau}\right] = \left[\hat{J}_{(1)}^{\nu\tau}\frac{\partial v_s^k}{\partial U^\tau}\right] = -\left[\hat{J}_{(0)}^{\nu\tau}\frac{\partial v_{s+1}^k}{\partial U^\tau}\right] \tag{3.6}$$

$$\left[\hat{R}_\xi^\nu\hat{J}_{(0)}^{\xi\tau}\frac{\partial h_{(1),s-1}^k}{\partial U^\tau}\right] = \left[\hat{J}_{(1)}^{\nu\tau}\frac{\partial h_{(1),s-1}^k}{\partial U^\tau}\right] = -\left[\hat{J}_{(0)}^{\nu\tau}\frac{\partial h_{(1),s}^k}{\partial U^\tau}\right] \tag{3.7}$$

according to Theorem 1.

Let us now multiply the equalities

$$\left[\hat{J}_{(1)}^{\xi\tau}\frac{\partial v_s^k}{\partial U^\tau}\right] = -\left[\hat{J}_{(0)}^{\xi\tau}\frac{\partial v_{s+1}^k}{\partial U^\tau}\right] \;, \;\; \left[\hat{J}_{(1)}^{\xi\tau}\frac{\partial h_{(1),s}^k}{\partial U^\tau}\right] = -\left[\hat{J}_{(0)}^{\xi\tau}\frac{\partial h_{(1),s+1}^k}{\partial U^\tau}\right]$$

by $\hat{\Omega}_{(0)\nu\xi}$ from the left. Since

$$\frac{\partial v_{s+1}^k}{\partial U^\tau}(z) = \frac{\partial h_{(1),s+1}^k}{\partial U^\tau}(z) = 0 \;, \;\; s \geq 0$$

we get again from the Theorem 1:

$$\frac{\partial v_{s+1}^k}{\partial U^\nu} = -\left[\hat{\Omega}_{(0)\nu\xi}\hat{J}_{(1)}^{\xi\tau}\frac{\partial v_s^k}{\partial U^\tau}\right] = -\left[\frac{\partial v_s^k}{\partial U^\tau}\hat{R}_\nu^\tau\right] \;, \;\; s \geq 0 \tag{3.8}$$

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$$\frac{\partial h^k_{(1),s+1}}{\partial U^\nu} = -\left[\hat{\Omega}_{(0)\nu\xi}\hat{J}^{\xi\tau}_{(1)}\frac{\partial h^k_{(1),s}}{\partial U^\tau}\right] = -\left[\frac{\partial h^k_{(1),s}}{\partial U^\tau}\hat{R}^\tau_\nu\right] \ , \ \ s \geq 0 \tag{3.9}$$

(action from the right).

Using the relation

$$\left[f_X D^{-1}\right] = -f_X$$

for the action from the right of D^{-1} on any $f(U)$ such that $f(z) = 0$ we get that the last six terms in the expression for $\hat{R}^\nu_\xi \left[\hat{R}^n\right]^\xi_\mu$ can be written as

$$-\sum_{q=1}^{N+g_0} E^q_{(0)}\left[\hat{J}^{\nu\tau}_{(0)}\frac{\partial v^q_1}{\partial U^\tau}\right]D^{-1}\left[\frac{\partial v^q_0}{\partial U^\xi}\left(\hat{R}^n\right)^\xi_\mu\right] - \sum_{q=1}^{g_1} e_{(1)q}\left[\hat{J}^{\nu\tau}_{(0)}\frac{\partial h^q_{(1),0}}{\partial U^\tau}\right]D^{-1}\left[\frac{\partial h^q_{(1),0}}{\partial U^\xi}\left(\hat{R}^n\right)^\xi_\mu\right] =$$

$$= (-1)^{n+1}\sum_{q=1}^{N+g_0} E^q_{(0)}\left[\hat{J}^{\nu\tau}_{(0)}\frac{\partial v^q_1}{\partial U^\tau}\right]D^{-1}\frac{\partial v^q_n}{\partial U^\mu} + (-1)^{n+1}\sum_{q=1}^{g_1} e_{(1)q}\left[\hat{J}^{\nu\tau}_{(0)}\frac{\partial h^q_{(1),0}}{\partial U^\tau}\right]D^{-1}\frac{\partial h^q_{(1),n}}{\partial U^\mu}$$

Using now the relations (3.6) and (3.7) we get the part (I) of the theorem.

(II) To avoid much calculations we just write that according to Theorem 2 we have the relations

$$E^k_{(0)}\frac{\partial v^q_{n-s}}{\partial U^\xi}\left[\hat{J}^{\xi\sigma}_{(0)}\frac{\partial v^k_0}{\partial U^\sigma}\right] \equiv \left(P^{qk}_{n-s,0}\right)_X = \frac{d}{dX}P^{qk}_{n-s,0} - P^{qk}_{n-s,0}\frac{d}{dX}$$

$$E^k_{(0)}\frac{\partial h^q_{(1),n-s}}{\partial U^\xi}\left[\hat{J}^{\xi\sigma}_{(0)}\frac{\partial v^k_0}{\partial U^\sigma}\right] \equiv \left(S^{qk}_{n-s,0}\right)_X = \frac{d}{dX}S^{qk}_{n-s,0} - S^{qk}_{n-s,0}\frac{d}{dX}$$

for some $P^{qk}_{n-s,0}(U)$, $S^{qk}_{n-s,0}(U)$, $P^{qk}_{n-s,0}(z) = 0$, $S^{qk}_{n-s,0}(z) = 0$ and so the expression $\hat{J}_{(n)} = \hat{R}^n\hat{J}_{(0)}$ can be written according on $\mathcal{F}_0(z)$ to (3.3) and (3.2) and Theorem 1 as

$$\hat{J}^{\nu\mu}_{(n)} = (\text{local part of } \hat{R}^n) \times (\text{local part of } \hat{J}_{(0)}) +$$

$$+(-1)^n\sum_{k=1}^{N+g_0} E^k_{(0)}\left(\sum_{s=1}^n\left[\hat{J}^{\nu\tau}_{(0)}\frac{\partial v^k_s}{\partial U^\tau}\right]\frac{\partial v^k_{n-s}}{\partial U^\xi}g^{\xi\mu}_{(0)}\right) +$$

$$+(-1)^n\sum_{k=1}^{g_1} e_{(1)k}\left(\sum_{s=1}^n\left[\hat{J}^{\nu\tau}_{(0)}\frac{\partial h^k_{(1),s-1}}{\partial U^\tau}\right]\frac{\partial h^k_{(1),n-s}}{\partial U^\xi}g^{\xi\mu}_{(0)}\right) +$$

$$+\sum_{q=1}^{N+g_0} E^q_{(0)}\left[\left(\hat{R}^n\right)^\nu_\xi \hat{J}^{\xi\tau}_{(0)}\frac{\partial v^q_0}{\partial U^\tau}\right]D^{-1}\left[\hat{J}^{\mu\sigma}_{(0)}\frac{\partial v^q_0}{\partial U^\sigma}\right] -$$

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$$-(-1)^n \sum_{k=1}^{N+g_0} E_{(0)}^k \left(\sum_{s=1}^{n} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial v_s^k}{\partial U^\tau} \right] D^{-1} \left[\hat{J}_{(0)}^{\mu\sigma} \frac{\partial v_{n-s}^k}{\partial U^\sigma} \right] \right) -$$

$$-(-1)^n \sum_{k=1}^{g_1} e_{(1)k} \left(\sum_{s=1}^{n} \left[\hat{J}_{(0)}^{\nu\tau} \frac{\partial h_{(1),s-1}^k}{\partial U^\tau} \right] D^{-1} \left[\hat{J}_{(0)}^{\mu\sigma} \frac{\partial h_{(1),n-s}^k}{\partial U^\sigma} \right] \right)$$

(since $\hat{J}_{(0)}$ is skew-symmetric it's action from the right differs by sign from the action from the left). Using now (3.6) we get the part (II) of the Theorem. Let us mention also that it is important that we consider the space $\mathcal{F}_0(z)$ to use the equality

$$D^{-1} \frac{d}{dX} \frac{\partial v_0^k}{\partial U^\xi} = \frac{\partial v_0^k}{\partial U^\xi}$$

for $k = 1, \ldots, N$.

(III) We have

$$\hat{\Omega}_{(-n)\nu\mu} = \hat{\Omega}_{(0)\nu\xi} \left(\hat{R}^n \right)_\mu^\xi$$

Using again the functions $P_{0s}^{qk}(U)$ and $Q_{0s}^{qk}(U)$ we can write

$$\hat{\Omega}_{(-n)\nu\mu} = \sum_{k=1}^{N+g_0} E_{(0)}^k \frac{\partial v_0^k}{\partial U^\nu} D^{-1} \left[\frac{\partial v_0^k}{\partial U^\xi} \left(\hat{R}^n \right)_\mu^\xi \right] +$$

$$+ (-1)^n \sum_{k=1}^{N+g_0} E_{(0)}^k \left(\sum_{s=1}^{n} \left[\hat{\Omega}_{(0)\nu\xi} \hat{J}_{(0)}^{\xi\tau} \frac{\partial v_s^k}{\partial U^\tau} \right] D^{-1} \frac{\partial v_{n-s}^k}{\partial U^\mu} \right) +$$

$$+ (-1)^n \sum_{k=1}^{g_1} e_{(1)k} \left(\sum_{s=1}^{n} \left[\hat{\Omega}_{(0)\nu\xi} \hat{J}_{(0)}^{\xi\tau} \frac{\partial h_{(1),s-1}^k}{\partial U^\tau} \right] D^{-1} \frac{\partial h_{(1),n-s}^k}{\partial U^\mu} \right)$$

Since

$$\frac{\partial v_s^k}{\partial U^\tau}(z) = 0 \ , \quad \frac{\partial h_{(1),s-1}^k}{\partial U^\tau} = 0 \ . \ s \geq 1$$

we get the part (III) using Theorem 1 and (3.6) on $L(\mathcal{M}^N, z)$.

Theorem is proved.

Let us mention also that if both $det\, g_{(0)}^{\nu\mu} \neq 0$ and $det\, g_{(1)}^{\nu\mu} \neq 0$ (and the form Q in non-degenerate on $\mathcal{W}$) then also the series of "negative" Hamiltonian operators $\hat{J}_{(-n)} = \hat{R}^{-n} \hat{J}_{(0)}$ and "positive" symplectic forms $\hat{\Omega}_{(n)} = \hat{\Omega}_{(0)} \hat{R}^{-n}$ will be weakly nonlocal. This situation takes place for example in the Hamiltonian structures of Whitham systems for KdV, NLS and SG hierarchies. The local bi-Hamiltonian structure for the averaged KdV hierarchy was constructed in [1] (see also [2]-[3]) using the (Dubrovin-Novikov) procedure of averaging of local field-theoretical brackets for Gardner-Zakharov-Faddeev and Magri brackets. Both metrics of the

corresponding DN-brackets are non-degenerate in this case (and there is no requirements on Q). Also in [2]-[3] the local bracket for the averaged "VG-equation"

$$\varphi_{tt} - \varphi_{xx} + V'(\varphi) = 0$$

which is the generalization of SG system using the same procedure was constructed. In [22] all the brackets for averaged KdV, NLS and SG hierarchies having local (DN) or constant curvature (MF) form were enumerated using a nice differential-geometrical approach. It can be shown that all the pencils represented in [1]-[3], [22] actually satisfy the requirements of Theorems 2 and 3. The recursion operator approach for the local bi-Hamiltonian structure for the averaged KdV hierarchy (in the diagonal form) was investigated in [23], [24] and all the "positive" operators $\hat{J}_{(n)}$ were explicitly found in [24] in this case. In [27]-[28] the general procedure of averaging of brackets (0.8) which gives the weakly non-local Hamiltonian operators for the averaged systems with Hamiltonian structure (0.8) was constructed. For many "integrable" systems this method also gives all the "positive" weakly nonlocal Poisson brackets of Ferapontov type for the corresponding Whitham hierarchy. However, we see here that also the "negative" Hamiltonian operators and Symplectic structures for the averaged KdV, NLS and SG should be weakly non-local according to Theorems 2, 3. We believe that there should be a good procedure of averaging of "negative" Hamiltonian operators for the integrable systems giving the brackets of Ferapontov Type and the general averaging procedure for the weakly non-local Symplectic Structures

$$\hat{\Omega}_{\nu\mu}(x,y) = \sum_{k=1}^{N} C_{\nu\mu}^{(k)}(\varphi, \varphi_x, \ldots)\delta^{(k)}(x-y) + \sum_{k,s=1}^{G} d_{ks}\frac{\delta H_{(k)}}{\delta\varphi(x)}\nu(x-y)\frac{\delta H_{(s)}}{\delta\varphi(y)}$$

giving the weakly non-local symplectic structures and the corresponding F-brackets for the Whitham systems.

We believe also that the general compatible F-brackets should be important for the integration of non-diagonalizable (bi-Hamiltonian) systems which can not be integrated by Tsarev's method.

Let us mention also that the requirement of non-degeneracy of form Q on $\mathcal{W}$ is also important in Theorem 3. Such, it is possible to show that the "negative" and "positive" Poisson structures corresponding to the pair of operators (2.16)-(2.17) will not be weakly nonlocal.

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